-3

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME V.

SESSION 1886-87.

EDINBURGH:
1887.

PRINTED BY

JOHN LINDSAY, HIGH STREET, EDINBURGH.

INDEX.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

FIFTH SESSION, 1886-87.

First Meeting, November 13th, 1886.

DR FERGUSON, F.R.S.E., President, in the Chair.

Dr FERGUSON, as retiring President, delivered an address, for which and for his kindness in granting the Society free accommodation in the Edinburgh Institution, he was accorded a hearty vote of thanks.

For this Session the following Office-Bearers were elected:—

President—Mr GEORGE THOM, M.A.

Vice-President—Mr W. J. MACDONALD, M.A., F.R.S.E.

Secretary—Mr A. Y. FRASER, M.A., F.R.S.E.

Treasurer—Mr JOHN ALISON, M.A.

Committee—

Messrs R. E. ALLARDICE, M.A.; R. M. FERGUSON, Ph.D., F.R.S.E.; GEORGE A. GIBSON, M.A.; WILLIAM HARVEY, B.A., LL.B.; J. S. MACKAY, M.A., F.R.S.E.; THOMAS MUIR, M.A., LL.D., F.R.S.E.

Solutions of Euclid's Problems, with a rule and one fixed aperture of the compasses, by the Italian geometers of the sixteenth century.

By J. S. Mackay.

Haec ac similia ad ostentationem ingenii, utilitatem vero pene nullam, inventa sunt.

Hieronymus Cardanus, *De Subtilitate*, Liber XV.

Chasles in his *Aperçu Historique sur l'origine et le développement des Méthodes en Géométrie* (seconde édition, 1875, pp. 214-215) makes the following statement:

"Essays of the same kind as the geometry of the rule and that of the compasses, and which hold, so to speak, the mean between the two, had long previously engaged the attention of famous mathematicians. Cardan first of all in his book *De Subtilitate* had resolved several of Euclid's problems by the straight line and a single aperture of the compasses, as if one had in practice only a rule and invariable compasses. Tartalea was not long in following his rival on this field, and extended this mode of treatment to some new problems. (*General trattato di numeri et misure; 5ta parte, libro terzo;* in-fol. Venise, 1560). Finally, a learned Piedmontese geometer, J.–B. de Benedictis, made it the object of a treatise entitled: *Resolutio omnium Euclidis problematum, aliorumque ad hoc necessario inventorum, una tantummodo circini data apertura;* in-4°. Venise, 1553."

As very little is known of these essays, and as the treatises in which they occur are far from being readily accessible, I have thought it worth while to make an abstract of them. This abstract, if its interest should prove to be historical rather than scientific, may afford some amusement to those who are fond of geometrical curiosities. I have confined myself to the solutions of the problems in Euclid's first six books; those of the problems in the remaining books are not given at all by Cardano, are very briefly dismissed by Benedetti, and in any case are now not worth resuscitation. With reference to the form in which the solutions are presented, it ought perhaps to be said that the sequence of the problems has in all the three cases been rigidly adhered to. The lettering of the figures has frequently been changed, most of the demonstrations have been omitted as unnecessary, such as have been inserted are not always those given by the authors, but the constructions have not been tampered with. When

the constructions are the same or substantially the same as Euclid's, they have not been reproduced. Before, however, proceeding to the solutions, a word or two must be said in rectification of the statement of Chasles.

In the early part of 1547 Lodovico Ferraro, who was then a public lecturer on mathematics in Milan, and had been a dependant of the celebrated physician and algebraist Hieronimo Cardano (Tartaglia is never tired of calling him "suo creato"), sent an impertinent challenge to Tartaglia, and a brisk war of pamphlets ensued between the two. Thirty-one questions were proposed by each to be answered by the other within a given time, and it was with Tartaglia's questions that the problem arose which forms the subject of the present paper. The questions were first published in the *Seconda Riposta data da Nicolo Tartalea Brisciano a Messer Lodovico Ferraro* (Venice, 21 April, 1547). They are published again, along with their solutions and with the solutions of twenty-two of the thirty-one questions set by Ferraro, in Tartaglia's *General Trattato di Numeri et Misure* (Venice, 1560), Fifth Part, Third Book. Here Tartaglia tells us how the research originated. "I set myself one day (in order not to be unemployed) to try if it was possible to resolve, with any aperture of the compasses proposed by an adversary, the 26th* proposition of the sixth book of Euclid, namely, that where it is proposed to describe a superficies similar to a given rectilineal superficies and equal to another, a thing which not only I soon assured myself was possible, but even found it to be possible to resolve (with such a condition) all his other geometrical problems to be worked on a plane, excepting those where it falls to describe a terminated circle (as is the case in the 4th, 5th, 8th, 9th, 13th, and 14th propositions of his fourth book, and likewise in the 25th and 33rd of the third)."

Ferraro's answers to Tartaglia's questions were first printed in a pamphlet, the heading of which begins, *Quinto Cartello di Lodovico Ferraro contra Messer Nicolo Tartaglia*, and the last part of which is entitled *Risolutione fatto per Lodovico Ferraro à i trentaun' quesiti mandatagli da risolvere per Messer Nicolo Tartaglia* (Milan, October, 1547).

---

* This is the 25th proposition of Euclid's sixth book. Tartaglia in his translation of Euclid's *Elements* into Italian (Venice, 1543) now and then changes the order of Euclid's propositions.

Cardano, in his work *De Subtilitate Libri XXI.* (Norimbergae, 1550) entitles the fifteenth book *De incerti generis aut inutilibus subtilitatibus.* He there, without naming Tartaglia, mentions that both Ferraro and himself found out in a few days how all that Euclid proves, (with the exception of the inscription and circumscription of circles), when the aperture of the compasses may be varied, could be shown by them when the aperture was any invariable one proposed by an adversary. He says that Ferraro printed the demonstration of all this, but as it was for a controversial purpose, he did not think it would survive, and in order that so rare an example of subtilty might not perish, he gives an account of it.

Last of all, Giovanni Battista Benedetti (whose name is Latinised into Joannes Baptista de Benedictis) took the matter up, and published the treatise mentioned by Chasles. In his verbose preface or dedication, Benedetti says nothing about the origin of the problem, or about the published solutions of Ferraro and Cardano.

Tartaglia's Solutions.

1. *On a given straight line to describe an equilateral triangle.*

Euclid I. 1. See figures 1, 2.

Let AB be the given straight line.

Make AC and BD each equal to the given aperture. On AC describe the equilateral triangle ACE, and on BD the equilateral triangle BDF. Let AE and BF intersect at G.

GAB is the equilateral triangle required.

2. *Through a given point to draw a straight line parallel to a given straight line.*

Euclid I. 31. See figures 3, 4.

Let A be the given point, BC the given straight line.

With A as centre cut BC at D. Join AD and produce it, making DE equal to AD. With E as centre cut BC at F. Join EF and produce it, making FG equal to EF. Join AG.

AG is the parallel required.

If the circle with A as centre and the given aperture as radius do not cut BC, choose a point H near enough to BC, and through it draw the parallel HK, as before. Then through A draw AG parallel to HK.

3. *To bisect a given angle.*

Euclid I. 9. See figure 5.

Let ACB be the given angle.

With C as centre cut CA and CB at D and E. With D and E as centres describe circles intersecting in F. Join CF.

CF is the bisector required.

4. *To bisect a given straight line.*

Euclid I. 10. See figure 6.

Let AB be the given straight line.

With A and B as centres describe circles cutting AB at C and D. If these circles do not intersect, with C and D as centres describe circles; and so on. Finally, let E and F be the points of intersection. Join EF cutting AB at G.

G is the point of bisection.

5. *To draw a perpendicular to a given straight line from a given point in it.*

Euclid I. 11. See figure 7.

Let AB be the given straight line, C the given point.

With C as centre cut CA and CB at D and E. Bisect CD and CE at G and H. With G and H as centres describe circles intersecting in F. Join CF.

CF is the perpendicular required.

6. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12. See figure 8.

Let AB be the given straight line, C the given point.

With C as centre cut AB at D and E. With D and E as centres describe circles intersecting in F. Join CF.

CF is the perpendicular required.

If the circle with C as centre and the given aperture as radius do not cut AB, draw a straight line near enough to C and parallel to AB. To that straight line draw a perpendicular from C, and produce the perpendicular to meet AB.

7. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2. See figures 9, 10.

(*a*) Let A be the given point, BC the given straight line. (Fig. 9)

Join AB, and through C draw CD parallel to AB. Through A draw AE parallel to BC meeting CD at E.

AE is the straight line required.

(*b*) Let A be the given point, BC the given straight line. (Fig. 10)

Join AB, and on it describe the equilateral triangle DAB. Produce DA and DB to E and F. With B as centre cut BC and BF at G and H. Join GH, and through C draw CK parallel to GH and meeting BF at K. Through K draw KL parallel to BA and meeting AE at L.

AL is the straight line required.

8. *From the greater of two given straight lines to cut off a part equal to the less.*

Euclid I. 3. See figures 11, 12, 13, 14, 15, 16.

(*a*) Let the given straight lines AB and CD, of which CD is the greater, be parallel. (Fig. 11)

Join AC, and through B draw BE parallel to AC, and meeting CD at E.

CE is the part required.

(*b*) Let the given straight lines AB and AD, of which AD is the greater, be joined at A. (Fig. 12)

With A as centre cut AD and AB at E and F. Join EF, and through B draw BG parallel to FE, and meeting AD at G.

AG is the part required.

(*c*) Let the given straight lines AB and AD, of which AD is the greater, be in the same straight line. (Fig. 13)

From A draw AE perpendicular to AB. From AE cut off AF equal to AB; and from AD cut off AG equal to AF.

AG is the part required.

(*d*) Let the given straight lines AB and CD, of which CD is the greater, be neither parallel, joined at a point, nor in the same straight line. (Fig. 14)

Through A draw AE parallel to CD. From AE cut off AF equal to AB; and from CD cut off CG equal to AF.

CG is the part required.

(*e*) Let the given straight lines AB and CD, of which CD is the greater, be in the same straight line but not contiguous. (Fig. 15)

From B and C draw BE and CF respectively perpendicular to AB and CD. Cut off BE equal to BA; through E draw EF parallel to AD and meeting CF at F. From CD cut off CG equal to CF.

CG is the part required.

(*f*) Let the given straight lines AB and AC, of which AC is the greater, be joined at A; and let it be required to cut off a part equal to AB from the end C. (Fig. 16)

Join BC, and on it describe the equilateral triangle DBC. Produce DB and DC to E and F. Cut off BE equal to BA; and through E draw EF parallel to BC and meeting DF at F. Cut off CG equal to CF.

CG is the part required.

9. *At a given point in a given straight line to make an angle equal to a given angle.*

Euclid I. 23. See figure 17.

Let A be the given point, AB the given straight line, and CDE the given angle.

With D as centre cut DE at F; with F as centre cut DC at G, and join FG. From AB cut off AH equal to DG; with A and H as centres describe circles intersecting in K. Join KA, KH.

HAK is the angle required.

When angle CDE is obtuse, make, as before, angle HAK equal to its supplement; and find the supplement of angle HAK.

10–14. Euclid I. 42, 44, 45, 46; II. 11.

15. *Given two squares, to apply to one of them a gnomon equal to the other.*

This is not one of Euclid's problems, but is given in Tartaglia's edition of *The Elements* as II. 14. See figure 18.

Let ABCD, EFGH be the two squares.

Produce AB, and cut off BK equal to EF. Join CK, and from AK cut off AL equal to CK. On AL describe the square ALMN.

BCDNML is the gnomon required.

16. Euclid VI. 10.

17. *To find a mean proportional between two given straight lines.*

Euclid VI. 13. See figure 19.

Let AB and BC be the two given straight lines.

Draw any straight line AD making an angle with AC, and cut off AE equal to the given aperture of the compasses. With E as centre describe a circle cutting AD again at F. Join CF, and through B draw BG parallel to CF and meeting AF at G. From G draw GH perpendicular to AF and meeting the circumference at H. Cut off GK equal to GH, and through K draw KL parallel to FC and meeting AC produced at L.

BL is the mean proportional required.

The following demonstration may be given:
On account of the parallels BG, LK, CF,

$$\begin{aligned} AB : BL : BC &= AG : GK : GF, \\ &= AG : GH : GF. \end{aligned}$$

Now GH is a mean proportional between AG and GF;
therefore BL is a mean proportional between AB and BC.

18. *To describe a square equal to a given triangle.*

This is a particular case of Euclid II. 14. It is given as II. 15 in Tartaglia's edition of *The Elements.*

Construct a rectangle equal to the given triangle, and find a mean proportional between its sides. The square on this mean proportional is the square required.

19. *To describe a square equal to a given rectilineal figure.*

Euclid II. 14. It is given by Tartaglia as the last proposition of the second book.

(*a*) The method of solution is the same as that of the preceding problem.

(*b*) Resolve the rectilineal figure into triangles. Find the side of a square equal to the first triangle. At one end of it raise a perpen-

dicular equal to the side of a square which is equal to the second triangle; and draw the hypotenuse of this right-angled triangle. At one end of the hypotenuse raise a perpendicular equal to the side of a square which is equal to the third triangle; and draw the hypotenuse of this right-angled triangle. Continue this process till all the triangles which make up the rectilineal figure are exhausted. The last hypotenuse is a side of the required square.

20. Euclid III. 1.

21. *From a given point to draw a tangent to a given circle.*

Euclid III. 17. See figure 20.

Let ABG be the given circle, and D the given point.

Find C the centre of the given circle, and draw the straight line DACB. Produce BD to E, and make DE equal to DA.
Divide BA at F, so that BF is to FA as BD to DE; and from F draw FG perpendicular to AB and meeting the circumference at G. Join DG.

DG is the tangent required.

The following demonstration may be given:

Because $$BF : FA = BD : DE,$$
$$= BD : DA;$$
therefore BA is divided harmonically at F and D.
Hence FG is the polar of D, and DG a tangent to the circle.

Tartaglia gives no proof of his construction, but refers to Apollonius's *Conics*, Book I., Prop. 34, and adds that in the same manner a tangent may be drawn to the three other conic sections.* He mentions that this problem was the first of the thirty-one questions proposed by him to Cardano and Ferraro in their public dispute.

22–26. Euclid III. 30, 34; VI. 11, 12, 9.

---

* Apollonius proves that this construction will give a tangent to the hyperbola, the ellipse, and the circle; but Prop. 35 shows that he knew how to modify it to obtain a tangent to the parabola.

27. *In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.*

Euclid IV. 1. See figure 21.

Let CDF be the given circle, and AB the given straight line.

Draw any diameter CD, and from it cut off CE equal to the third proportional to CD and AB. From E draw EF perpendicular to CD and meeting the circumference at F. Join CF.

CF is the chord required.

28–38. Euclid IV. 2, 3, 6, 7, 10, 11, 12, 15, 16; VI. 18, 25.

The last of these problems was the second of Tartaglia's questions.

39. *To find a square equal to the difference of two given squares.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows. See figure 22.

Let ABCD, EFGH be the given squares, the latter being the greater.

Take a straight line LM equal to twice the given aperture, and on it describe a semicircle. Find a fourth proportional to EF, AB, and LM; in the semicircle place a chord LK equal to this fourth proportional; and join KM. Lastly, find PQ a fourth proportional to LM, KM, and EF.

PQ is a side of the square required.

40. *To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22. See figure 23.

Let AB, CD, EF be the three given straight lines.

Take a straight line HK equal to CD. Find a square equal to the sum of the squares on AB and CD; then find a second square equal to the difference between this square and the square on EF. On HK describe a rectangle equal to the half of the second square; and cut off HL equal to the breadth of the rectangle. Find a third square equal to the difference between the squares on AB and HL; from L draw LG perpendicular to HK and equal to a side of this third square; and join GH, GK.

GHK is the triangle required.

Tartaglia adds that to ensure the perpendicular always falling inside the triangle, the longest side should be chosen for base.

The demonstration follows from Euclid II. 12 or 13.

41. Euclid VI. 28.

This was the third of Tartaglia's questions.

42. *To describe a parallelogram which shall be similar to a given parallelogram and equal to another parallelogram and a triangle.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows.

43. Euclid VI. 29.

This was the fourth of Tartaglia's questions.

44. Euclid VI. 30.

Ferraro's and Cardano's Solutions.

1. Euclid I. 9. Tartaglia's solution.

2. Euclid I. 10. Tartaglia's solution.

3. Euclid I. 11. Tartaglia's solution.

4. *From the greater of two unequal straight lines drawn from the same point to cut off a part equal to the less.*

Particular case of Euclid I. 3. See figure 24.

Let AB and AC be the given straight lines, AC being the greater.

Bisect the angle BAC by AD. With B as centre describe a circle cutting AD at E; with E as centre describe a circle cutting AC at F.

AF is the part required.

If the circle with B as centre does not meet AD, bisect the angles BAD, CAD, and repeat the preceding construction.

5. *On a given straight line to describe an isosceles triangle.*

This is not one of Euclid's problems, but is inserted as subsidiary

to the problem which follows. The construction is effected by the 2nd and 3rd problems.

6. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2. See figure 25.

Let A be the given point, BC the given straight line.

Join AB, and on it describe the isosceles triangle ABD. Produce DB and DA; cut off BE equal to BC, and DF equal to DE.

AF is the straight line required.

7. Euclid I. 3.

8. *At a given point in a given straight line to make an angle equal to a given angle.*

Euclid I. 23. Tartaglia's solution.

If the aperture is too small, which will be the case when the given angle is obtuse, bisect the given angle; at the given point make an angle equal to half the given angle; and repeat the construction.

9. *On a given straight line to describe an equilateral triangle.*

Euclid I. 1.

Let AB be the given straight line.

With the given aperture describe an equilateral triangle; and at A and B make angles equal to two of the angles of this triangle.

10. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12.

Let AB be the given straight line, C the given point.

Through C draw CD parallel to AB; and from C draw CE perpendicular to CD.

11. *To find a mean proportional between two given straight lines.*

Euclid VI. 13. See figure 26.

Let AC and CB be the given straight lines.

Place AC and CB in the same straight line. At A draw AF perpendicular to AB and equal to twice the given aperture. Join BF; and through C draw CE parallel to BF. Make CG equal to EF, and CK equal to AE. On KG, which is equal to twice the given aperture, describe the semicircle KLG; Draw CL perpendicular to KG, and meeting the circumference at L. Join GL; and through B draw BM parallel to GL.

CM is the mean proportional required.

The following demonstration may be given:

Because	AE : AC = EF : CB,
	= CG : CB,
	= CL : CM;
therefore	AE : CL = AC : CM;
therefore	KC : CL = AC : CM.
Again	CL : CG = CM : CB;
and	KC : CL = CL : CG;
therefore	AC : CM = CM : CB.

12. Euclid II. 14. Tartaglia's solution.

13. *To draw a tangent to a circle from a given external point.*

Euclid III. 17. See figure 27.

Let BGC be the given circle, A the external point.

Through O, the centre of the circle BGC, draw the secant ABC; and find DE a mean proportional between AB and AC. Draw EF perpendicular to DE, and equal to the radius OB; and join DF. At O make the angle AOG equal to the angle DFE; and join AG.

AG is the tangent required.

14–15. Euclid III. 25, 34.

16. *On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.*

Euclid III. 33. See figure 28.

Let AB be the given straight line, C the given angle.

With the given aperture describe a circle DEF; and from it cut off a segment DFE containing an angle equal to C. Take any point F in the arc of the segment, and join FD, FE. At A and B make angles BAG, ABG respectively equal to the angles EDF, DEF.

Then G is a point on the arc of the segment described on AB, and as many other points as may be necessary can be found in a similar manner.

17. *In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.*

Euclid IV. 1. See figure 29.

Let O be the centre of the given circle, CD the given straight line.

Draw any radius OA; and find FG a fourth proportional to OA, CD, and the given aperture. On FG describe an isosceles triangle EFG, whose sides EF, EG are each equal to the given aperture. At O make angle AOB equal to angle FEG; and join AB.

AB is the chord required.

18. *To construct a triangle whose sides shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22. See figure 30.

Let A, B, C be the three given straight lines, and let A be greater than B, and B greater than C.

With the given aperture describe a circle, and through O its centre draw the diameter DG. Take OE such that $A : B = DO : OE$; and EF such that $B : C = OE : EF$.
Again take HK such that $FE : EG = DE : HK$; and produce HK to L, so that KL may be equal to FE. In the circle place the chord MN equal to HL; from MN cut off MP equal to KL; and join OP. Lastly, at the extremities of A make angles respectively equal to angles OMP and MOP.

The following demonstration may be given:

(*a*) To prove that the point F lies outside the circle.

A, B, and C are proportional to DO, OE, and EF. But B and C are greater than A; therefore OE and EF are greater than DO.

(*b*) To prove that DG is greater than HL.

$DE : HK = FE : EG$, by construction.
Now DE is greater than EG;
and DE is greater than FE, because $DE : FE = A + B : C$;
and DE is greater than HK, because FE is greater than EG,

Hence of the four proportionals DE, HK, FE, EG, the greatest is DE, and consequently the least is EG; therefore DE + EG is greater than HK + FE, that is DG is greater than HL.

(*c*) To prove that MO, OP, PM are respectively equal to DO, OE, EF, and therefore proportional to A, B, C.

MO and DO are radii of the same circle; and PM was made equal to KL or EF. It remains therefore to prove OP equal to OE.

$$\begin{aligned} &\text{Because} & \mathrm{DE} : \mathrm{HK} &= \mathrm{FE} : \mathrm{EG}; \\ &\text{therefore} & \mathrm{DE} \cdot \mathrm{EG} &= \mathrm{HK} \cdot \mathrm{FE}; \\ && &= \mathrm{HK} \cdot \mathrm{KL}, \\ && &= \mathrm{NP} \cdot \mathrm{PM}, \\ && &= \mathrm{RP} \cdot \mathrm{PQ}. \end{aligned}$$

Now DG = RQ; therefore DE = RP, and OE = OP.
Hence the triangle MOP is similar to the triangle whose sides are A, B, C.

19. *To describe an isosceles triangle having each of the base angles double of the vertical angle.*

Euclid IV. 10.

Take any straight line AB, and divide it at C so that $AB \cdot BC$ is equal to AC^2. Construct a triangle whose sides shall be equal or proportional to AB, AB, and AC.

Cardano adds the two following problems:

20. *A diameter of a circle is given, and a point in it. To raise from the given point a perpendicular of such a length that it shall just meet the circumference.*

The length of the perpendicular is equal to a mean proportional between the segments of the diameter.

21. *On a given hypotenuse to construct a right-angled triangle having one of its sides equal to a given straight line.*

See figure 31.

Let AB be the hypotenuse, CD the side.

With the given aperture describe the circle EFG, and draw a diameter EF. To AB, CD, EE find a fourth proportional; in the circle EFG place FG equal to this fourth proportional; and join EG.

At B make angle ABH equal to angle EFG; cut off BH equal to CD; and join AH.

ABH is the triangle required.

BENEDETTI'S SOLUTIONS.

1. *To draw a perpendicular to a given straight line from a given point in it.*

Euclid I. 11. See figure 32.

Let AB be the given straight line, C the given point.

With C as centre describe the semicircle DFE. On CD and CE construct the equilateral triangles CDF, CGE; and join FG. On FG construct the equilateral triangle FHG; and join CH.

CH is the perpendicular required.

2. *To produce a given straight line, which is less than the given aperture, so that the part produced may be equal to the given straight line.*

See figure 33.

Let AB be the given straight line.

At B draw BD perpendicular to AB. With A as centre cut BD at E; and with E as centre cut AB produced at F.

BF is the part required.

3. *To do the same thing when the given straight line is greater than the given aperture.*

See figure 34.

Let AB be the given straight line.

From AB cut off consecutively distances equal to the given aperture, till DB the part that remains is less than the given aperture. Produce DB to E so that BE may be equal to DB. From AE produced, beginning at E, cut off as many distances equal to the given aperture as have been cut off from AB.

4. Euclid I. 10. Tartaglia's solution.

5. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12. See figure 35.

Let AB be the given straight line, C the given point.

Join BC, and produce it so that CD may be equal to BC. Join DA, and bisect it at E. Join CE, and from C draw CF perpendicular to CE.

CF is the perpendicular required.

6. *Through a given point to draw a straight line parallel to a given straight line.*

Euclid I. 31.

Let AB be the given straight line, C the given point.

From B draw BD perpendicular to AB; and from C draw CE perpendicular to BD.

CE is the parallel required.

7. *From a given point to draw a straight line equal and parallel to a given straight line.*

Let AB be the given straight line, C the given point.

Join CB. Through C draw CD parallel to AB, and through A draw AD parallel to BC.

CD is the straight line required.

8. *To cut off from the greater of two given straight lines a part equal to the less, or to produce the less till it is equal to the greater.*

Euclid I. 3. See figure 36.

Let AB and CD be the two given straight lines.

From C draw CE equal and parallel to AB. With C as centre describe a circle cutting CE and CD or these lines produced at F and G. Join FG; and through E draw EH parallel to FG, and meeting CD or CD produced at H.

CH is the straight line required.

9. *To bisect a given angle.*

Euclid I. 9. See figure 37.

Let BAC be the given angle.

With A as centre describe a circle cutting AB and AC at D and E. Join DE; bisect DE at F; and join AF.

AF is the bisector required.

10. Euclid I. 23. Tartaglia's solution.

11. Euclid I. 42.

12. *On a given [indefinite] straight line to construct a triangle equal and similar to a given triangle.*

See figure 38.

Let AB be the given straight line, CDE the given triangle.

Cut off BF equal to DC; and make angle FBG equal to angle CDE. Cut off BG equal to DE; and join FG.

FBG is the triangle required.

13–14. Euclid I. 44, 46.

15. *Given two squares, to apply to one of them a gnomon equal to the other.*

Tartaglia's solution.

16–17. Euclid II. 11; VI. 10.

18. Euclid VI. 13. Tartaglia's solution.

19. *To describe a square equal to a given triangle.*

Tartaglia's solution.

20. Euclid III. 1.

21. *In a given circle to place a chord equal to one given straight line, which is less than the diameter, and parallel to another given straight line.*

See figure 39.

Let EKF be the given circle, AB the straight line to which the chord is to be equal, CD the straight line to which the chord is to be parallel.

Through O the centre of the given circle draw the diameter EF

parallel to OD; cut off OG and OH each equal to half of AB. Draw GK and HL perpendicular to EF; and join KL.

KL is the chord required.

22. *With three given straight lines, two of which are equal to each other, and are together greater than the third, to construct a triangle.*

See figure 40.

Let A, B, C be the three given straight lines, of which A and B are equal.

Take any straight line DEF, making DE equal to A, and EF equal to C; from D draw DG equal to the given aperture. Join EG, and through F draw FH parallel to EG. In a circle whose centre is O, and which is described with the given aperture as radius, place the chord KL equal to GH; and join OK, OL. Make OM, ON equal to A, B; and join MN.

OMN is the triangle required.

23. *To construct a triangle equal to a given triangle, and such that it may have an angle equal to a given angle, and a side equal to a given straight line.*

The construction is effected by the 13th problem.

24. *From a given point to draw a tangent to a given circle.*

Euclid III. 17. See figure 41.

Let DBE be the given circle, A the given point.

Find O the centre of the circle; and join AO cutting the circumference at B. Divide BO at C so that $BC : CO = AB : BO$; from C draw CD perpendicular to AO and cutting the circumference at D; and join AD.

AD is the tangent required.

The following demonstration may be given:

Because $\quad BC : CO = AB : BO,$

therefore $\quad BO : CO = AO : BO;$

therefore $\quad AO \cdot CO = BO^2.$

Hence CD is the polar of A, and AD a tangent to the circle.

25–26. Euclid III. 30, 34.

27. *To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22.

Benedetti discusses (in six pages) five cases of this problem.

(1) When two of the sides are equal.

(2) When the three sides are equal.

(3) When the square on one side is equal to the sum of the squares on the two other sides.

(4) When the square on one side is less than the sum of the squares on the two other sides.

(5) When the square on one side is greater than the sum of the squares on the two other sides.

In the first case the problem becomes the 22nd; in the second case the 36th; in the third case it reduces to that of constructing a right-angled triangle, having given the sides containing the right angle. The solutions of the fourth and fifth cases, depending on Euclid II. 13 and 12, are substantially the same as Tartaglia's. Benedetti concludes with the remark: "Et contra illos omnes excellentissimos Mathematicos priscos modernosque qui dixerunt impossibile esse hoc problema alio modo posse concludi quam ut docet xxii primi Euclidis, ego vero deo dante labente Anno diuinae incarnationis MDLII Die xv Octobris illud inueni."

28–34. Euclid IV. 2, 3, 6, 7, 10, 11, 12.

35. *In a given circle to inscribe a regular hexagon.*

Euclid IV. 15.

Find O the centre of the circle, and draw the diameter AOD. In the circle place the chord BC equal to the radius and parallel to AD. Join BO, CO, and produce them to meet the circumference in E, F. Join AB, CD, DE, EF, FA.

ABCDEF is the hexagon required.

36. Euclid I. 1. Ferraro's solution.

37. *From a given point in the circumference of a given circle to draw a chord which shall be equal to a given straight line.*

Euclid IV. 1. See figure 42.

Let AB be the given straight line. C the given point.

Find O the centre of the given circle, and draw the diameter COD. In the circle place the chord EF equal to AB, and parallel to CD. Draw the diameter FOG; and through C draw CH parallel to FG.

CH is the chord required.

38–45. Euclid IV. 26; VI. 11, 12, 9, 18, 25, 28, 29, 30.

46. *On a given hypotenuse to construct a right-angled triangle having one of its sides equal to a given straight line.*

Cardano's solution.

47. *Given two straight lines, one of which is less than half of the other, to divide the greater into two segments such that the smaller straight line shall be a mean proportional between them.*

See figure 43.

Let AB and BC be the two straight lines, BC being less than half of AB.

Place BC at right angles to AB, and join C with O the middle point of AB. With O as centre describe the semicircle DGE; from E draw EF perpendicular to AB and meeting CO at F. Through F draw FG parallel to AB and meeting the semicircle at G; from G draw GH perpendicular to AB. Lastly, divide AB at K so that AK : KB = DH : HE.

AK and KB are the required segments.

The following demonstration may be given:

$$\begin{aligned} EF : BC &= EO : BO, \\ &= 2EO : 2BO, \\ &= DE : AB. \end{aligned}$$

Now EF, which is equal to HG, is a mean proportional between DH and HE, the segments of DE;

therefore BC is a mean proportional between the segments of AB, when AB is divided similarly to DE;

that is, AB is a mean proportional between AK and KB.

48. Euclid IV. 5.

49. *To construct a triangle similar to a given triangle, and such that the centre of its circumscribed circle may be at a given point and the radius of it equal to a given straight line.*

See figure 44.

Let ABC be the given triangle, D the given point, and E the given radius.

Find F the centre of the circle circumscribed about triangle ABC; and join FA, FB, FC. From D draw DK equal to E; on DK construct triangle DKG similar to triangle FCA, and triangle DKH similar to triangle FCB; and join GH.

GHK is the triangle required.

50–54. Euclid IV. 4, 8, 9, 13, 14.

55. *About a given centre to describe a regular figure which shall be similar to a given regular figure and shall have a given circumscribed radius.*

The method of solution is similar to that of 49.

56. Euclid III. 33.

57. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2.

Let A be the given point, BC the given straight line.

From A draw an indefinite straight line AD, and from AD cut off AE equal to BC.

58. *To find the centre of a circle, an arc of which is given.*

Euclid III. 25.

Draw any two chords not parallel to each other. The straight lines which bisect these chords perpendicularly will meet at the required centre.

59. *To find the diameter of a circle which shall be equivalent to two given circles.*

Make the diameters of the two given circles the base and perpendicular of a right-angled triangle. The hypotenuse of this triangle will be the diameter of the required circle.

Second Meeting, December 10*th*, 1886.

GEORGE THOM, Esq., President, in the Chair.

Composition de Mathématiques Élémentaires proposée au Concours d' Agrégation de 1886.

Solution par M. PAUL AUBERT.

On donne un cercle et deux points P et Q situés sur un diamètre, on joint les points P et Q aux extrémités A et B d'un diamètre du cercle par les droites PA et QB qui se coupent au point M. On fait tourner le diamètre AB et on demande

I. D' étudier les variations du rapport $\frac{MA}{MB}$, *et de construire la figure quand le rapport a une valeur donnée.*

II. D' étudier les variations de l' angle AMB, et de construire la figure quand cet angle a une valeur donnée.

III. A' et B' étant les seconds points d' intersection des droites MA, MB avec la circonférence donnée, trouver le lieu du centre du cercle circonscrit au triangle MA'B'.

I. On a (fig. 45)

$$\frac{\mathrm{MA}}{\mathrm{MB}}=\frac{\sin\mathrm{ABM}}{\sin\mathrm{BAM}}.$$

Menons par le point B la parallèle BR à MP, on a

$$\frac{\sin\mathrm{ABM}}{\sin\mathrm{BAM}}=\frac{\sin\mathrm{OBQ}}{\sin\mathrm{OBR}}=\frac{\mathrm{OQ}\sin\mathrm{BOQ}}{\mathrm{BQ}}:\frac{\mathrm{OR}\sin\mathrm{BOQ}}{\mathrm{BR}}=\frac{\mathrm{OQ}}{\mathrm{OR}}\cdot\frac{\mathrm{BR}}{\mathrm{BQ}}.$$

Appelons a et b les distances des points P et Q au centre de la circonférence donnée dont nous désignerons le rayon par R, et soit λ l' angle BOQ du diamètre mobile AB avec le diamètre fixe PQ. On a

$$\frac{\mathrm{MA}}{\mathrm{MB}}=\frac{b}{a}\,\frac{\mathrm{BR}}{\mathrm{BQ}}=\frac{b}{a}\sqrt{\frac{a^2+\mathrm{R}^2-2a\mathrm{R}\cos\lambda}{b^2+\mathrm{R}^2-2b\mathrm{R}\cos\lambda}}.$$

Si les points P et Q, au lieu d' être de part et d' autre du point O, sont du même côté de ce point (fig. 46), il est facile de voir que

les relations précédentes subsistent, à condition de changer sous le radical a en $-a$. On est donc amené à étudier les variations de l'expression

$$y=\frac{a^2+R^2-2aR\cos\lambda}{b^2+R^2-2bR\cos\lambda}$$

quand λ varie de 0° à 360°. Il suffira de faire varier λ de 0° à 180°, puisque cos λ prend toutes ses valeurs possibles dans cet intervalle, puis de prendre pour l' intervalle de 180° á 360° les valeurs de l' expression déjà trouvées mais dans l' ordre inverse.

Quand cos λ varie de $+1$ à -1 les deux termes de la fraction y restent toujours positifs. Par suite y ne devient jamais nulle ni infinie. Supposons d' abord P et Q de part et d' autre du centre. On peut écrire

$$\begin{aligned} y &= \frac{a}{b}\,\frac{b(a^2+R^2)-2abR\cos\lambda}{a(b^2+R^2)-2abR\cos\lambda} \\ &= \frac{a}{b}\left[1+\frac{b(a^2+R^2)-a(b^2+R^2)}{a(b^2+R^2)-2abR\cos\lambda}\right] \\ &= \frac{a}{b}\left[1+\frac{(R^2-ab)(b-a)}{a(b^2+R^2)-2abR\cos\lambda}\right] \end{aligned}$$

Quand cos λ varie de $+1$ à -1 le dénominateur de la fraction augmente constamment. Si son numérateur est positif, y diminue ; si ce numérateur est négatif, y augmente continuellement. Le rapport $\frac{MA}{MB}$ varie de $\frac{b}{a}\,\frac{R-a}{R-b}$ à $\frac{b}{a}\,\frac{R+a}{R+b}$ toujours dans le même sens, la première expression étant prise en valeur absolue.

Si les points P et Q sont du même côté du centre, on a

$$\frac{MA}{MB}=\frac{b}{a}\sqrt{\frac{a^2+R^2+2aR\cos\lambda}{b^2+R^2-2bR\cos\lambda}}.$$

Quand cos λ varie de $+1$ à -1, le numérateur de la fraction diminue, son dénominateur augmente. Donc le rapport diminue continuellement. Il varie de

$$\frac{b}{a}\,\frac{a+R}{b-R} \text{ à } \frac{b}{a}\,\frac{a-R}{b+R}.$$

Dans les deux cas le rapport $\frac{MA}{MB}$ varie toujours dans un certain sens quand λ croît de 0° à 180°, et dans le sens contraire quand λ continue à croître de 180° à 360°.

Construction géométrique : La construction du point M dépend

de la position du diamètre AB, qui est elle même déterminée par la position du point B. Or nous avons vû que

$$\frac{BR}{BQ} = \frac{a}{b}\,\frac{MA}{MB}.$$

Si le rapport $\frac{MA}{MB}$ est donné, on connaît donc la valeur du rapport $\frac{BR}{BQ}$. Pour obtenir le point B, on déterminera d' abord les deux points I et I′ du diamètre QR (fig. 47) dont le rapport des distances aux points Q et R a la valeur $\frac{a}{b}\,\frac{MA}{MB}$, puis on décrira sur II′ comme diamètre une circonférence qui coupera la circonférence donnée en deux points répondant à la question, si la valeur donnée pour le rapport $\frac{MA}{MB}$ est comprise entre les valeurs limites trouvées précédemment. On peut retrouver ces valeurs limites en discutant les conditions de possibilité de la construction géométrique.

II. Étude de l' angle AMB.

On a dans le triangle BQR dont l' angle B est egal à l' angle AMB

$$\overline{QR}^2 = \overline{BQ}^2 + \overline{BR}^2 - 2BQ \cdot BR\cos B,$$

d' où

$$\cos B = \frac{\overline{BQ}^2 + \overline{BR}^2 - \overline{QR}^2}{2BQ \cdot BR}.$$

Mais

$$\overline{BQ}^2 = R^2 + b^2 - 2bR\cos\lambda,$$
$$\overline{BR}^2 = R^2 + a^2 - 2aR\cos\lambda,$$
$$\overline{QR}^2 = (a-b)^2.$$

Effectuant ces substitutions, il vient

$$\cos B = \frac{R^2 + ab - (a+b)R\cos\lambda}{\sqrt{(R^2+b^2)(R^2+a^2) - 2(a+b)(ab+R^2)R\cos\lambda + 4abR^2\cos^2\lambda}}.$$

Il est avantageux de se servir de l' expression de tgB. On trouve sans difficulté en partant de la formule précédente

$$\operatorname{tg}B = \frac{(a-b)R\sin\lambda}{R^2 + ab - (a+b)R\cos\lambda}.$$

Posons $\operatorname{tg}\frac{\lambda}{2} = t$, il vient

$$\operatorname{tg}B = \frac{2(a-b)Rt}{(R+a)(R+b)t^2 + (R-a)(R-b)}.$$

Nous nous servirons de cette expression pour étudier les variations de l' angle B quand λ varie de 0° à 360°, c'est-à-dire, quand t varie de zéro à $+\infty$ et de $-\infty$ à zéro. Il suffit de faire varier t de zéro à $+\infty$, car quand il varie de $-\infty$ à zéro, tg B reprend les mêmes valeurs dans l' ordre inverse mais changées de signe.

Pour cela considérons la fonction $y = \dfrac{mt}{pt^2 + q}$.

Si on écrit la relation qui lie y à t sous la forme

$$pyt^2 - mt + qy = 0$$

on voit qu' y ne pourra prendre que les valeurs satisfaisant à l' inégalité

$$m^2 - 4pqy^2 > 0.$$

Deux cas à distinguer 1° $pq < 0$.

L' inégalité est toujours verifiée et y peut prendre toutes les valeurs de $-\infty$ à $+\infty$.

2° $pq > 0$.

Posons $pq = k^2$. L' inégalité s' écrit

$$(m - 2ky)(m + 2ky) > 0$$

et, m et k désignant des quantités positives, on doit avoir

$$-\frac{m}{2k} < y < \frac{m}{2k}.$$

Dans ce cas y reste compris entre deux limites correspondant aux valeurs de t fournies par la relation précédente, où l' on donne successivement à y les deux valeurs $-\dfrac{m}{2k}$ et $+\dfrac{m}{2k}$, et en prenant chaque fois pour t la demi-somme des racines de l' equation ainsi obtenue

$$t = -\frac{k}{p} \text{ et } t = +\frac{k}{p}.$$

En résumé, si nous ne considérons que les valeurs positives de t, les variations de y correspondantes sont représentées pour le premier cas par la courbe (fig. 48), et pour le second cas par la courbe (fig. 49), si on a $p > 0$, et par la courbe (fig. 50) si on a $p < 0$.

Revenons maintenant à l' étude de tg B et appliquons les résultats que nous venons d' obtenir.

La condition relative au signe de pq conduit à étudier le signe de $(R + a)(R + b)(R - a)(R - b)$. Cette expression ne change pas quand on y remplace a par $-a$. Donc dans tous les cas de figure elle est du signe de $(R - a)(R - b)$.

1°. $(R - a)(R - b) < 0$, ce qui exprime que l' un des points est

intérieur et l' autre extérieur à la circonférence. Dans ce cas l' angle B varie de 180° à 90°, puis diminue jusqu' à 0° ; il est droit pour

$$\operatorname{tg}\frac{\lambda}{2} = \sqrt{\frac{(a-\mathrm{R})(\mathrm{R}-b)}{(\mathrm{R}+a)(\mathrm{R}+b)}} \quad \text{ou} \quad \cos\lambda = \frac{\mathrm{R}^2+ab}{(a+b)\mathrm{R}},$$

comme l' indiquait la valeur de cos B.

2°. $(\mathrm{R}-a)(\mathrm{R}-b)>0$. Les deux points sont alors tous deux extérieurs ou tous deux intérieurs.

Si on a $(\mathrm{R}+a)(\mathrm{R}+b)>0$, ce qui exclut le cas où les points seraient extérieurs et du même côté, l' angle B est toujours aigu, il croît de zéro à une valeur maxima donnée par

$$\operatorname{tg}\mathrm{B} = \frac{(a-b)\mathrm{R}}{\sqrt{(\mathrm{R}^2-a^2)(\mathrm{R}^2-b^2)}}$$

et correspondante à $\operatorname{tg}\frac{\lambda}{2} = \sqrt{\frac{(\mathrm{R}-a)(\mathrm{R}-b)}{(\mathrm{R}+a)(\mathrm{R}+b)}}$.

Si on a $(\mathrm{R}+a)(\mathrm{R}+b)<0$, ce qui exprime que les deux points sont extérieurs à la circonférence et du même côté, l' angle B est toujours obtus, il varie de 180° à un minimum pour augmenter ensuite jusqu' à 180°.

En résumé, quand les points sont l' un intérieur l' autre extérieur à la circonférence, l' angle B varie toujours dans le même sens et prend toutes les valeurs de 0° à 180°.

Quand les points sont tous deux intérieurs, ou tous deux extérieurs l' angle B est toujours soit aigu soit obtus ; dans le premier cas, il commence par croître jusqu' à un certain maximum pour décroître ensuite jusqu' à zéro ; dans le second cas, il décroît de 180° à un minimum qu' il dépasse ensuite pour retourner à la valeur 180°.

Pour construire géométriquement le point M répondant à une valeur donnée de l' angle M, il suffit de déterminer la position du point B. L' angle QBR étant connu, et les points Q et R étant fixes, on décrira sur RQ un segment de cercle capable de l' angle donné. Les points d' intersection de ce segment avec la circonférence donnée donnent les points B cherchés. On voit que si les points sont l'un intérieur et l'autre extérieur, il y aura toujours au-dessus du diamètre QR un point B répondant à la question. Si les points sont tous deux intérieurs ou tous deux extérieurs, il pourra y avoir deux positions, une seule, ou aucune suivant les cas. L' étude des conditions de possibilité de cette construction géométrique du point B conduirait aux résultats établis plus haut par une autre voie.

III. Les triangles semblables PMQ et RBQ donnent

$$\frac{BR}{PM}=\frac{QR}{QP}=\frac{QB}{QM};$$

on, puisque $BR = PA$,

$$\frac{PA}{PM}=\frac{QR}{QP}.$$

Le dernier rapport est constant, puisque les points Q et R sont fixes. Le point P étant fixe, il résulte de cette égalité que le point M est sur une circonférence homothétique à la circonférence donnée par rapport au point P, qui est centre d' homothétie directe.

On a aussi $$\frac{QB}{QM}=\frac{QR}{QP}.$$

Cette égalité montre que le point Q est le centre d' homothétie inverse des deux circonférences.

Ces résultats se rapportent à la figure 45; ils seraient renversés pour le cas de la figure 46. Dans tous les cas P et Q sont les centres d' homothétie directe et inverse de la circonférence donnée O, et de la circonférence lieu du point M, que nous appellerons circonférence O′. Considérons par exemple la figure 51. Les points M et A′ sont anti-homologues. La tangente en A′ au cercle O et la tangente en M à la circonférence O′ font donc avec MA′ des angles égaux, et se coupent en I sur l' axe radical des deux cercles. Pour la même raison, les points M et B′ étant anti-homologues, la tangente en B′ au cercle O rencontre MI sur l' axe radical, c'est-à-dire au point I, et les angles IMB′ et IB′M sont égaux. Le point I est donc situé sur les perpendiculaires élevées au milieu de MA′ et au milieu de MB′; c' est donc le centre du cercle circonscrit au triangle MA′B′. Il résulte des constructions précédentes que ce point est toujours sur l' axe radical des circonférences O et O′. Donc le lieu demandé est cet axe radical.

The Equilateral and the Equiangular Polygon.

By R. E. Allardice, M.A.

THE EQUILATERAL POLYGON.

Since an n-gon is determined by $2n-3$ conditions, and $n-1$ conditions are involved in its being equilateral, there are still in the case of an equilateral n-gon $n-2$ conditions to be determined. These $n-2$ conditions cannot all be given in terms of the angles, since an

infinite number of n-gons may always be described similar to, but not necessarily congruent with, any given equilateral n-gon. Hence only $n-3$ of the angles of an equilateral n-gon may be assigned arbitrarily, and there must therefore be 3 independent relations connecting the angles of any equilateral n-gon. These three conditions may be obtained by projecting the perimeter of the n-gon on any three lines.

Let a be the common length of the sides; A, B, C......L, the exterior angles of the n-gon.

Projection on one of the sides adjacent to the angle A gives the relation

$$a\cos A + a\cos(A+B) + a\cos(A+B+C) + \ldots\ldots = 0;$$

$$\therefore \quad \cos A + \cos(A+B) + \cos(A+B+C) + \ldots\ldots = 0. \qquad (1).$$

Similarly projection on the other sides gives

$$\cos B + \cos(B+C) + \cos(B+C+D) + \ldots\ldots = 0, \qquad (2),$$

$$\cos C + \cos(C+D) + \cos(C+D+E) + \ldots\ldots = 0, \text{ \&c.}; \qquad (3),$$

and projection on lines perpendicular to these gives

$$\sin A + \sin(A+B) + \sin(A+B+C) + \ldots\ldots = 0, \qquad (4),$$

$$\sin B + \sin(B+C) + \sin(B+C+D) + \ldots\ldots = 0. \qquad (5).$$

There thus arise $2n$ equations; but, as has been seen, only three of these are independent. This may be proved analytically in the following manner.

Assume equations (1), (2), and (4) above, and assume also

$$\sin B + \sin(B+C) + \sin(B+C+D) + \ldots\ldots = p. \qquad (6).$$

Let $\cos A + i\sin A = e^{iA} = \alpha$, $\cos B + i\sin B = e^{iB} = \beta$, &c.;

then $\cos(A+B) + i\sin(A+B) = e^{i(A+B)} = e^{iA}e^{iB} = \alpha\beta$.

Hence equations (1), (2), (4), and (6) are equivalent to

$$\alpha + \alpha\beta + \alpha\beta\gamma + \ldots\ldots + (\alpha\beta\gamma\ldots\ldots\lambda) = 0,$$

$$\beta + \beta\gamma + \beta\gamma\delta + \ldots\ldots\ldots (\beta\gamma\delta\ldots\ \ldots\alpha) = pi$$

$$\therefore \quad \alpha - (\alpha^2\beta\gamma\ldots\ \ldots\lambda) = -\alpha pi;$$

but $\alpha = \cos A + i\sin A$ is not equal to 0,

$$\therefore \quad 1 - (\alpha\beta\gamma\ldots\ldots\lambda) = -pi,$$

$$\therefore \quad \cos(A+B+\ldots\ldots L) + i\sin(A+B+\ldots\ldots L) = 1 + pi,$$

$$\therefore \quad \cos(A+B+\ldots\ldots L) = 1, \ \sin(A+B+\ldots\ldots L) = p,$$

$$\therefore \quad A+B+C\ldots\ldots L = 2n\pi, \ p = 0, \text{ and } \alpha\beta\ldots\ldots\lambda = 1.$$

Hence $\beta + \beta\gamma + \beta\gamma\delta + \ldots\ldots(\beta\gamma\delta\ldots\ldots\lambda\alpha) = 0.$

Hence also, since β is not equal to 0,

$$1 + \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha) = 0,$$

that is, $(\alpha\beta\gamma\ldots\ldots\lambda) + \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha) = 0,$

$$\therefore \quad \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha\beta) = 0;$$

and in the same way the truth of all the other relations may be established.

These equations may also be written in the form

$$1+\beta+\beta\gamma+\ldots\ldots(\beta\gamma\ldots\ldots\lambda)=0, \text{ \&c.}$$

In order now to prove that the equations do in general involve three independent conditions, it will be sufficient to show that in the case where n is 3, they are sufficient to determine the actual values of A, B and C.

The equations may be assumed in the form

$$\cos A+\cos(A+B)+\cos(A+B+C)=0,$$
$$\cos B+\cos(B+C)+\cos(B+C+A)=0,$$
$$A+B+C=2n\pi;$$

which lead at once to the equations

$$1+\cos A+\cos C=0,$$
$$1+\cos B+\cos A=0;$$

whence also $\quad 1+\cos C+\cos B=0.$

These equations give $\cos A=\cos B=\cos C=-\frac{1}{2}$; the only solution of which lying between 0 and π is $A=B=C=2\pi/3$.

From the equations in the case of the quadrilateral may be deduced the equations $\cos A=\cos C$ and $\cos B=\cos D$. The only equilateral quadrilateral in the ordinary sense is of course the rhombus; but the above equations include the limiting cases of two straight lines which meet at a point taken twice over, and a single straight line taken four times over.

It should be noticed that in general it is not allowable to take the angles in an arbitrary order. This is exemplified in the case of the quadrilateral; but it may be shown that in general two angles cannot change places.

Suppose that an equilateral polygon may be formed with certain angles taken in the order ABCDEF......and also with the same angles taken in the order EBCDAF......

Then $\quad \alpha+\alpha\beta+\alpha\beta\gamma+\alpha\beta\gamma\delta+\alpha\beta\gamma\delta\epsilon+\ldots\ldots=0,$

and $\quad \epsilon+\epsilon\beta+\epsilon\beta\gamma+\epsilon\beta\gamma\delta+\epsilon\beta\gamma\delta\alpha+\ldots\ldots=0$

$\therefore \quad (\alpha-\epsilon)(1+\beta+\beta\gamma+\beta\gamma\delta)=0.$

Hence the angles cannot change places unless they are equal, or the intervening angles satisfy the relation $1+\beta+\beta\gamma+\beta\gamma\delta=0$. This condition involves that the sides between the two vertices considered, themselves form a closed polygon; for it implies that the sum of the projections of these sides on each of two straight lines at right angles

to one another is zero. This equation which is equivalent to two conditions, since it involves the imaginary unit, only contains $(n-1)$ of the angles in the general case, and is therefore not in itself sufficient to determine that the n angles may be the angles of an equilateral polygon.

It may be noted further that three conditions of the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$

are not sufficient; for they may be satisfied by angles whose sum is not a multiple of 2π. For let n equal lines P_1P_2, $P_2P_3\ldots\ldots P_nP_{n+1}$, be drawn making angles A, B, C......K with one another (the first $n-1$ of the angles); then the first condition will be satisfied if P_{n+1} lies in the perpendicular to P_1P_2 through the point P_1. Let now another straight line $P_{n+1}P_{n+2}$ be drawn making with P_nP_{n+1} an angle L; then the second condition will be satisfied if the point P_{n+2} lies in the perpendicular to P_2P_3 through the point P_2; and so on for a third, fourth, etc. condition, the next line making with that last considered an angle A, the next again an angle B, and so on. Thus in the case where n is 3, the conditions are satisfied by the angles of a square, that is, by the values $A=B=C=\pi/2$; in the case where n is 4, the four conditions of the form considered are satisfied by the angles of a regular hexagon, that is, by the values, $A=B=C=D=\pi/3$; and, in the general case, the n conditions are satisfied by the angles of a regular polygon of $2(n-1)$ sides, that is, by the values $A=B=C=\ldots\ldots=\pi/(n-1)$.

The three conditions may be given in the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$
$$1+\cos B+\cos(B+C)+\ldots\ldots=0$$
$$A+B+C+\ldots\ldots=2\pi.$$

Assume
$$1+\sin A+\sin(A+B)+\ldots\ldots=p$$
$$1+\sin B+\sin(B+C)+\ldots\ldots=q.$$

Then
$$1+\alpha+\alpha\beta+\ldots\ldots=pi$$
$$1+\beta+\beta\gamma+\ldots\ldots=qi$$
$$\alpha\beta\gamma\ldots\ldots=1.$$

$$\therefore\quad pi-qi\alpha=0;$$
$$\therefore\quad pi-qi(\cos A+i\sin A)=0;$$
$$\therefore\quad q\sin A+(p-q\cos A)i=0.$$

$\therefore$ if sin A is not equal to 0, $q=0$ and $p=0$.

If the equations are assumed in the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$
$$1+\sin A+\sin(A+B)+\ldots\ldots=0$$
$$A+B+C+\ldots\ldots=2\pi,$$

or in the equivalent form

$$1+\alpha+\alpha\beta+\ldots\ldots=0$$
$$\alpha\beta\gamma\ldots\ldots=1,$$

the truth of all the other equations is at once obvious.

From the above it is seen that when the conditions are given in any of the forms discussed here, either two of the three conditions must be derived from the projection of the sides of the polygon on two different straight lines, or one of them must be that the sum of the exterior angles is $2m\pi$, where m is some integer.

If n angles be found which may be made the angles of an equilateral polygon, then these angles may be combined in various ways so as to form the angles of other equilateral polygons. Suppose, for example, that n is 5; then the conditions connecting the angles may be written

$$1+\alpha+\alpha\beta+\alpha\beta\gamma+\alpha\beta\gamma\delta=0$$
$$\alpha\beta\gamma\delta\epsilon=1;$$

which may be written in the form

$$1+(\alpha\beta)+(\alpha\beta)(\gamma\delta)+(\alpha\beta)(\gamma\delta)(\epsilon\alpha)+(\alpha\beta)(\gamma\delta)(\epsilon\alpha)(\beta\gamma)=0$$
$$(\alpha\beta)(\gamma\delta)(\epsilon\alpha)(\beta\gamma)(\delta\epsilon)=1;$$

and these equations show that an equilateral polygon may be made of which the exterior angles are, A + B, C + D, E + A, B + C, D + E. In the same way, if n is not divisible by 3, the angles may be added together in sets of three to form the angles of a new equilateral polygon. And it may easily be seen that the angles may be combined in various ways. For let a radius vector rotate from the position OX through an angle A into the position OA, then through an angle B into the position OB, and so on. Now if the radius vector be conceived to start from OX and to rotate into the position of any of the lines just considered, OP say, then form OP into another position, OQ say, and so on, the different angles through which it rotates may be made the angles of an equilateral polygon, it being supposed that the radius vector always rotates in the same direction and never stops in the same position twice. This may be proved in the same way as the particular cases given above; but it is seen even more easily as a consequence of the laws of the composition of vectors.

THE EQUIANGULAR POLYGON.

In the fact that an n-gon is equiangular $(n-1)$ conditions are involved; and since the other $(n-2)$ conditions may all be given in terms of the sides, it follows that there must be two relations connecting the sides of an equiangular polygon.

The exterior angle of a regular n-gon is $2\pi/n$ or $2m\pi/n = \mathrm{A}$, say, where A may be assumed not to be a multiple of π; and hence α is not real and cannot vanish, where

$$\alpha = e^{\mathrm{A}i} = \cos\mathrm{A} + i\sin\mathrm{A}.$$

Let the sides be denoted by $a, b, c \ldots\ldots k$; and let the perimeter be projected on the side a and on a line perpendicular to a.

Then $$a + b\cos\mathrm{A} + c\cos 2\mathrm{A} + \ldots\ldots + k\cos(n-1)\mathrm{A} = 0;$$
$$b\sin\mathrm{A} + c\sin 2\mathrm{A} + \ldots\ldots + k\sin(n-1)\mathrm{A} = 0.$$

$$\therefore \quad a + b\alpha + c\alpha^2 + \ldots\ldots + k\alpha^{n-1} = 0; \qquad (1)$$
$$\alpha^n = 1.$$

Multiply equation (1) by α^{n-1}; then

$$b + c\alpha + d\alpha^2 + \ldots\ldots a\alpha^{n-1} = 0;$$

and in the same way all the other similar equations may be deduced.

It may easily be shown that the two equations may be assumed in the form

$$a + b\cos\mathrm{A} + c\cos 2\mathrm{A} + \ldots\ldots = 0;$$
$$b + c\cos\mathrm{A} + d\cos 2\mathrm{A} + \ldots\ldots = 0.$$

The two conditions may be expressed in terms of the sides alone as follows:—

If the sides are represented by $a_1, a_2, a_3 \ldots\ldots a_n$, the conditions are

$$a_1 + a_2\alpha + a_3\alpha^2 + \ldots\ldots + a_n\alpha^{n-1} = 0$$
$$a_2 + a_3\alpha + a_4\alpha^2 + \ldots\ldots + a_1\alpha^{n-1} = 0$$
$$\ldots\ldots\ldots\ldots$$
$$a_n + a_1\alpha + a_2\alpha^2 + \ldots\ldots + a_{n-1}\alpha^{n-1} = 0.$$

Hence

$$\alpha = -\begin{vmatrix} a_1 & a_3 & \ldots\ldots & a_n \\ a_2 & a_4 & \ldots\ldots & a_1 \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & a_1 & \ldots\ldots & a_{n-2} \end{vmatrix} \div \begin{vmatrix} a_2 & a_3 & \ldots\ldots & a_n \\ a_3 & a_4 & \ldots\ldots & a_1 \\ \ldots & \ldots & \ldots & \ldots \\ a_n & a_1 & \ldots\ldots & a_{n-2} \end{vmatrix}.$$

But α is not real, while $a_1, a_2, \ldots\ldots a_n$, are all real; and therefore both numerator and denominator in the value of α must vanish. It is obvious that all the other first minors of the circulant determinant $(a_1a_2\ldots\ldots a_n)$ must also vanish. The conditions may be expressed in the form—

$$\begin{vmatrix} a_2 & a_3 & \ldots\ldots & a_n & a_1 \\ a_3 & a_4 & \ldots\ldots & a_1 & a_2 \\ \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ a_n & a_1 & \ldots\ldots & a_{n-2} & a_{n-1} \end{vmatrix} = 0.$$

Conversely, if the condition

$$a_1 + a_2 a + a_3 a^2 + \ldots\ldots a_n a^{n-1} = 0$$

be given, where $a^n = 1$, it may be shown that an equiangular polygon can be formed with the sides $a_1, a_2 \ldots\ldots a_n$. For if a polygon be formed with these sides and with $(n-1)$ exterior angles, each equal to $2\pi/n$, the above condition shows that the polygon will be a closed one, and the condition $a^n = 1$ that the last exterior angle will also be $2\pi/n$.

In the reasoning of last paragraph a, instead of being $\cos 2\pi/n + i\sin 2\pi/n$, may have any one of the values $\cos 2k\pi/n + i\sin 2k\pi/n$, where k has any integral value from 0 to $(n-1)$. It would thus appear that there are n distinct species of equiangular n-gons, distinguished by the magnitude of the exterior angle; but some of these are degenerate cases and others are not distinct.

Thus if $$a = \cos 0 + i\sin 0$$
then $$a_1 + a_2 + \ldots\ldots + a_n = 0\,;$$
and since none of the sides are negative each must be zero.

Again if n is even and

$$a = \cos\pi + i\sin\pi,$$

the condition becomes

$$a_1 - a_2 + \ldots\ldots + a_{n-1} - a_n = 0\,;$$

and the polygon is a flat one, such as ABCDEFGHA (fig. 52).

Further the two values

$$a = \cos 2r\pi/n + i\sin 2r\pi/n,$$
$$a = \cos 2(n-r)\pi/n + i\sin 2(n-r)\pi/n,$$

give the same polygon, the angles regarded as the exterior angles in the two cases being the conjugates of one another.

All the other polygons are distinct; and hence it follows that if n is odd there are $\frac{1}{2}(n-1)$, and if n is even $\frac{1}{2}(n-2)$, equiangular n-gons of different species; all of which, of course, with the exception of one, are crossed polygons.

A particular case of the above is that in which the polygons are regular. In this case, however, if n is not prime, some of the polygons consist of those with a smaller number of sides taken several times over. Thus one of the regular octagons consists of a square

taken twice over. These may be called degenerate cases; but there are always certain regular n-gons which are not degenerate cases; one such being the ordinary regular non-crossed n-gon. The number of non-degenerate regular polygons is easily seen to be half the number of special roots of the equation $x^n - 1 = 0$. Now if p, q, r...... are primes, and if $n = p^\alpha q^\beta r^\gamma$......, the number of special roots of this equation is

$$n(1 - 1/p)(1 - 1/q)......$$

Hence the number of regular n-gons is

$$\tfrac{1}{2}n(1 - 1/p)(1 - 1/q)......$$

where p, q, r...... are the prime factors of n, including n if n be prime.

The number of regular n-gons may also be seen very easily by consideration of a circle divided into n equal parts. Each point may be joined to the next, or to the next but one, or to the next but two, and so on; and a regular n-gon will be formed in each case provided every point of division is included.

If an equiangular n-gon can be formed with the sides, a_1, $a_2 ...... a_n$ taken in a definite order, then an equiangular polygon of any other species may be formed by taking the same sides in a different order, namely in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$... provided neither of the polygons be of a species which degenerates when it becomes regular. For if an equiangular polygon of the kind considered can be formed with the sides $a_1, a_2 ...... a_n$, taken in that order, then

$$a_1 + a_2\alpha + ... + a_n\alpha^{n-1} = 0,$$

where α is a special root of the equation $x^n - 1 = 0$. If β is any other special root of this equation, then

$$\beta = \alpha^r,\ \beta^2 = \alpha^{2r},\ \&c.,$$

$$\therefore \qquad a_1 + a_{r+1}\beta + a_{2r+1}\beta^2 + ... = 0;$$

which shows that an equiangular polygon may be formed with the same sides taken in the order, a_1, a_{r+1}, a_{2r+1} ... Figs. 53, 54, 55, represent the three species of equiangular nonagons which can be formed by taking the same lines in different orders. The first is the ordinary non-crossed nonagon the exterior angle of which is $2\pi/9$; the second is formed from the first by taking every second side, and has $4\pi/9$ for its exterior angle; and the third is formed from the first by taking every fourth side, and has $8\pi/9$ for its exterior angle. It is in general impossible to form an equiangular nonagon of the

remaining (third) species with the same sides, as this species degenerates when it becomes regular, becoming in that case an equilateral triangle taken thrice over. The same thing is indicated by the fact that it is impossible to get all the sides of a nonagon by going round it and taking every third side.

The propositions of last paragraph may also be proved by means of the laws of composition of vectors. For if vectors be drawn through any point parallel to the sides of an equiangular polygon, these vectors will make equal angles with one another, and if they be compounded in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$...... they will again form an equiangular polygon, provided all the lines are included, when they are compounded in this way, that is, provided n be not a multiple of r.

It may now be shown that conversely an equiangular polygon may be formed with the sides $a_1, a_2 \ldots\ldots a_n$, if the matrix of the circulant $C(a_1a_2\ldots\ldots a_n)$ vanishes.

It is sufficient to show that if this matrix vanishes, then

$$a_1 + a_2\alpha + \ldots\ldots + a_n\alpha^{n-1} = 0, \text{ where } \alpha^n = 1.$$

Now if the above matrix vanishes, then the circulant $C(a_1a_2\ldots\ldots a_n)$ or this divided by $(a_1 + a_2 + \ldots\ldots + a_n)$ also vanishes.

But $$C(a_1a_2\ldots\ldots a_n) = \Pi(a_1 + a_2\alpha_r + \ldots\ldots + a_n\alpha_r^{n-1}),$$

where α_r is a root of $x^n - 1 = 0$, and r has every value from 1 to n.

Hence $$a_1 + a_2\alpha + \ldots\ldots + a_n\alpha^{n-1} = 0,$$

where α is some root of $x^n - 1 = 0$; and therefore an equilateral triangle may be formed with the sides taken in the above order, the exterior angle being one of the values of $2k\pi/n$.

It is obvious that the sides may not be taken in any arbitrary order and that in general only one of the angles $2k\pi/n$ may be taken. As a matter of fact, if the above-mentioned matrix vanishes, at least two of the factors $(a_1 + a_2\alpha_r + \ldots\ldots + a_n\alpha_r^{n-1})$ must vanish; namely those in which the quantities α_r are conjugate imaginaries. A possible case when n is even, included in the above, is that of the flat polygon mentioned before. This case arises when α is equal to π and the factor which vanishes is then $(a_1 - a_2 + a_3 - \ldots\ldots)$.

The condition that $a_1, a_2 \ldots\ldots a_n$ be the sides of an equiangular polygon may be represented as follows:—

$$\text{Mat.}C(a_1a_2\ldots\ldots a_n) = 0.$$

Now, if the sides taken in the order $a_1, a_{r+1}, a_{2r+1} \ldots\ldots$, form an equiangular polygon, which they will do if r is not a factor of n, then

$$\text{Mat.}C(a_1a_{r+1}a_{2r+1}\ldots\ldots) = 0.$$

Hence if one of these matrices vanishes so must the other. In fact it may easily be proved by means of the identity

$$C(a_1a_2\ldots\ldots a_n) = \Pi(a_1 + a_2\alpha + a_3\alpha^2\ldots\ldots)$$

that the two circulants are equal; that is the circulant $C(a_1a_2\ldots\ldots a_n)$ is not altered if the letters be written in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$ &c., if r be not a factor of n.

It follows from the above that if

$C(a_1a_2\ldots\ldots a_n) = 0$, and $a_1 + a_2 + \ldots\ldots + a_n$ is not equal to 0,

then $$\text{Mat.}C(a_1a_2\ldots\ldots a_n) = 0.$$

For in this case, for some value of α

$$a_1 + a_2\alpha + \ldots\ldots + a_n\alpha^{n-1} = 0;$$

and therefore, as has been seen before, $\text{Mat.}C(a_1a_2\ldots\ldots a_n) = 0$.
It is assumed that if n is even $a_1 - a_2 + a_3 - \ldots\ldots$ is not equal to 0; and that $a_1, a_2\ldots\ldots a_n$ are all real.

Again, if three consecutive sides be increased by x, $-2x\cos A$, x, the polygon still remains equiangular, A being the exterior angle of the polygon. Hence if the first matrix of a circulant vanish and if three consecutive letters p, q, r, be changed to $p+x$, $p-2x\cos A$, $r+x$, where x is arbitrary and A is a certain one of the angles $2k\pi/n$, the circulant will still vanish.

This may also be proved analytically; for let the factor of $C(a_1a_2\ldots\ldots a_n)$ which vanishes be

$$a_1 + a_2\alpha + \ldots\ldots a_n\alpha^{n-1}$$

where $$\alpha = \cos 2p\pi/n + i\sin 2p\pi/n.$$

Then $$a_{r+1}\alpha^r + a_{r+2}\alpha^{r+1} + a_{r+3}\alpha^{r+2}$$

becomes $$a_{r+1}\alpha^r + a_{r+2}\alpha^{r+1} + a_{r+3}\alpha^{r+2} + \alpha^r(x - 2x\alpha\cos 2p\pi/n + x\alpha^2);$$

and $$x - 2x\alpha\cos 2p\pi/n + x\alpha^2$$

$$= x\{1 - 2(\cos 2p\pi/n + i\sin 2p\pi/n)\cos 2p\pi/n + \cos 4p\pi/n + i\sin 4p\pi/n\}$$
$$= x\{(1 - 2\cos^2 2p\pi/n + \cos 4p\pi/n) + i(\sin 4p\pi/n - \sin 4p\pi/n)\}$$
$$= 0.$$

Conversely this transformation will indicate which of the factors of $C(a_1a_2\ldots\ldots a_n)$ vanishes when the circulant itself vanishes.

Particular cases. In the simpler cases the general conditions given above reduce to the following:—

Triangle $\quad a_1 = a_2 = a_3.$

Quadrilateral $\quad a_1 = a_3;\ a_2 = a_4.$

Pentagon. $\quad \{4a - (b+c+d+e) + \sqrt{5}(b-c-d+e)\} = 0;$

$\{(b-e)^2 + (d-c)^2\} + \sqrt{5}\{(b-e)^2 - (d-c)^2\} = 0.$

By means of these equations it may easily be seen that an equiangular pentagon can only have its sides commensurable if it be regular.

Hexagon.
$$a_1 + a_2 = a_4 + a_5 ;$$
$$a_2 + a_3 = a_5 + a_6 .$$

Octagon,
$$(a_2 - a_4 - a_6 + a_8) + \sqrt{2}(a_1 - a_5) = 0 ;$$
$$(a_2 + a_4 - a_6 - a_8) + \sqrt{2}(a_3 - a_7) = 0.$$

In an equiangular octagon with commensurable sides, opposite sides are equal.

Decagon,
$$4(a_1 - a_6) + (a_2 - a_3 + a_4 - a_5 - a_7 + a_8 - a_9 + a_{10}) + \sqrt{5}(a_2 + a_3 - a_4 - a_5 - a_7 - a_8 + a_9 + a_{10}) = 0 ;$$
$$(a_2 + a_5 - a_7 - a_{10})^2 + (a_3 + a_4 - a_8 - a_9)^2 + \sqrt{5}\{(a_3 + a_4 - a_8 - a_9)^2 - (a_2 + a_5 - a_7 - a_{10})^2\} = 0.$$

In an equiangular decagon with commensurable sides,
$$a_1 - a_6 = a_7 - a_2 = a_3 - a_8 = a_9 - a_4 = a_5 - a_{10} ;$$
where a_1 and a_6, a_2 and a_7, &c., are opposite sides.

Dodecagon.
$$2(a_1 - a_7) + (a_3 - a_5 - a_9 + a_{11}) + \sqrt{3}(a_2 - a_6 - a_8 + a_{12}) = 0 ;$$
$$2(a_4 - a_{10}) + (a_2 + a_6 - a_8 - a_{12}) + \sqrt{3}(a_3 + a_5 - a_9 - a_{11}) = 0.$$

In an equiangular dodecagon with commensurable sides,
$$a_2 - a_8 = a_{10} - a_4 = a_6 - a_{12} ; \; a_1 - a_7 = a_9 - a_3 = a_5 - a_{11} ;$$
where a_2 and a_8, a_4 and a_{10}, &c., are opposite sides.

Third Meeting, January 14th, 1887.

W. J. MACDONALD, Esq., M.A., Vice-President, in the Chair.

On Certain Inverse Roulette Problems.

By PROFESSOR CHRYSTAL.

The problem of designing cams or centrodes to produce any given motion in one plane is one of some practical importance; and it seems worth while to illustrate by examples some simple methods by which the solution can in certain cases be arrived at. These methods are founded, for the most part, on the use of the so-called Pedal Equation (or *p-r*-equation), which has great advantages in the present investigation, inasmuch as it depends on the form but not on the position of the curve which it represents.

Few of the results arrived at are absolutely new. Most of them have been found directly by Clerk Maxwell in a paper published in the Transactions of the Royal Society of Edinburgh, Vol. XVI., 1849.

If we suppose a plane Π to slide upon a fixed plane Π', it is obvious that the motion of Π is determined if the space loci of two points of Π, say P and Q, be given. There will therefore be an infinite number of ways of causing Π to move so that any point P in it may have a given locus. In other words, the problem to generate a given plane curve as a roulette is indeterminate.

I. If, however, there be given a fixed curve C′, then it is a determinate problem to find what curve C in Π must roll on C′ in order that the point P in Π may trace a given curve R.

II. Again, if there be given a curve C in Π, then it is a determinate problem to find a curve C′ such that, if C roll on C′, then P shall trace a given curve R.

Owing to the fact that the point P is fixed in Π, and thus affords an origin of reference for the curve C, or body centrode, as it is called, the first of these problems is in general easier than the second. We can at all events in general find, without much difficulty, an equation connecting the radius vector (r) from P and the perpendicular (p) from P on the tangent to C.

In fact, if K (Fig. 56) be the point of contact of the body and space centrodes C and C′, then, if P be the corresponding position of the point which traces out the given curve R, we know that PK is normal to R. Moreover, the tangent to C′ at K is the tangent to C at K. Hence, if PM be perpendicular to this tangent, we have (P being fixed with reference to C) $PK = r$, $PM = p$.

Now the two curves C′ and R are given in the plane Π'; and from their properties we can deduce a relation between PM, and PK, P being a variable point on R, and PK normal to R.

If this relation be $f(\text{PM}, \text{PK}) = 0$, then the p-r-equation to the curve C, with respect to P as origin, will be

$$f(p, r) = 0 \qquad (1).$$

Since $1/p^2 = 1/r^2 + (dr/r^2 d\theta)^2$, (1) gives us the differential equation

$$f\left\{\left(\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right)^{-\frac{1}{2}}, \ r\right\} = 0 \qquad (2).$$

The obtaining of the relation (1) is a comparatively simple matter in many cases; but as a rule the integration of the equation (2) presents great difficulty.

CASES WHERE THE SPACE CENTRODE IS A STRAIGHT LINE.

In such cases, if we take the given straight line as the x-axis, we have merely to find an equation between the normal PG and the ordinate PN to the given curve R; and this relation will furnish at once the p-r-equation to the body centrode, if we put $\text{PN}=p$, and $\text{PG}=r$.

Generation of a circle, the space centrode being any straight line. (Fig. 57).

Let a be the radius of the circle, b the distance of its centre A from the given straight line O′. We have

$$p=\text{PN}=b+a\sin\phi,$$
$$r=\text{PG}=a+b\operatorname{cosec}\phi,$$

ϕ being the angle PGD.

Hence
$$(p-b)(r-a)=ab,$$

or
$$\frac{a}{r}+\frac{b}{p}=1. \qquad (1).$$

From (1) we deduce

$$\frac{a^2}{r^2}-\frac{2a}{r}+1=b^2\left(\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right).$$

Hence
$$d\theta=\pm\frac{bdr}{r^2\sqrt{\left(1-\frac{2a}{r}+\frac{a^2-b^2}{r^2}\right)}} \qquad (2).$$

If $a>b$ we have

$$d\theta=\mp\frac{b}{\sqrt{a^2-b^2}}\frac{d\left(\frac{1}{r}-\frac{a}{a^2-b^2}\right)}{\sqrt{\left\{\left(\frac{1}{r}-\frac{a}{a^2-b^2}\right)^2-\frac{b^2}{(a^2-b^2)^2}\right\}}},$$

$$\chi=\frac{\sqrt{a^2-b^2}}{b}\theta+\alpha=\log\left\{\frac{1}{r}-\frac{a}{a^2-b^2}+\sqrt{\left(\frac{1}{a^2-b^2}-\frac{2a}{(a^2-b^2)}r+\frac{1}{r^2}\right)}\right\},$$

$$e^{\chi}-\left(\frac{1}{r}-\frac{a}{a^2-b^2}\right)=\sqrt{(\&c.)},$$

$$e^{2\chi}-2\left(\frac{1}{r}-\frac{a}{a^2-b^2}\right)e^{\chi}+\left(\frac{1}{r}-\frac{a}{a^2-b^2}\right)^2=\frac{1}{r^2}-\frac{2a}{(a^2-b^2)r}+\frac{1}{a^2-b^2},$$

$$\frac{1}{2}\left(e^{\chi}+\frac{b^2}{(a^2-b^2)^2}e^{-\chi}\right)=\frac{1}{r}-\frac{a}{a^2-b^2}.$$

This we may write

$$\frac{1}{2}e^{\alpha}\left(e^{c\theta/b}+\frac{b^2e^{-2\alpha}}{(a^2-b^2)^2}e^{-c\theta/b}\right)=\frac{1}{r}-\frac{a}{c^2},$$

where $c=\sqrt{a^2-b^2}$.

Since α depends merely on the choice of the prime radius vector through P, we may so select the latter that $b^2e^{-2\alpha}/(a^2-b^2)^2=1$, we then have

$$\frac{1}{2}\frac{b}{c^2}\left(e^{c\theta/b}+e^{-c\theta/b}\right)=\frac{1}{r}-\frac{a}{c^2},$$

$$r=\frac{c^2}{a+b\cosh(c\theta/b)}.$$

If $a<b$, (2) may be written

$$d\theta=+\frac{b}{\sqrt{b^2-a^2}}\frac{d\left(\frac{1}{r}+\frac{a}{b^2-a^2}\right)}{\sqrt{\left\{\frac{b^2}{(b^2-a^2)^2}-\left(\frac{1}{r}+\frac{a}{b^2-a^2}\right)^2\right\}}}.$$

Hence

$$\alpha+\frac{c\theta}{b}=\cos^{-1}\left(\frac{\frac{c^2}{r}+a}{b}\right);$$

where $c=\sqrt{b^2-a^2}$.

We may choose the prime radius so as to annul α; hence we have

$$b\cos\frac{c\theta}{b}=\frac{c^2}{r}+a,$$

or

$$r=\frac{c^2}{b\cos(c\theta/b)-a}.$$

If $a=b$, (2) becomes

$$d\theta=+\frac{ad\left(\frac{1}{r}\right)}{\sqrt{\left(1-\frac{2a}{r}\right)}},$$

$$\theta+\alpha=\sqrt{\left(1-\frac{2a}{r}\right)};$$

which, by properly choosing the prime radius, we may write

$$r=\frac{2a}{1-\theta^2}.$$

Generation of a straight line, the space centrode being a straight line inclined at an angle α to the given straight line. (Fig. 58).

Here we have

$$PG\cos\alpha=PN;$$

$$r\cos\alpha=p.$$

Hence the body centrode is an equiangular spiral whose pole is , and whose angle is $\frac{1}{2}\pi - a$.

Generation of an ellipse, the space centrode being the major axis. (Fig. 59).

If the co-ordinates of P be (x, y), so that $y^2 = (1-e^2)(a^2 - x^2)$,

we have $$p^2 = y^2 = (1-e^2)(a^2 - x^2);$$

and $$\begin{aligned} r^2 &= \mathrm{PG}^2 = \mathrm{NG}^2 + \mathrm{PN}^2, \\ &= (1-e^2)^2x^2 + y^2, \\ &= (1-e^2)(a^2 - e^2x^2). \end{aligned}$$

Hence $$r^2 - e^2p^2 = (1-e^2)^2a^2;$$

that is $$p^2 = \frac{a^2}{a^2-b^2}r^2 - \frac{b^4}{a^2-b^2} \qquad (1).$$

Now (see Williamson Diff. Calc. p. 346) if α be the radius of the fixed, β the radius of the rolling circle of an epicycloid, and if the centre of the fixed circle be the origin, the p-r-equation is

$$p^2 = \frac{(\alpha+2\beta)^2}{4\beta(\alpha+\beta)}r^2 - \frac{\alpha^2(\alpha+2\beta)^2}{4\beta(\alpha+\beta)} \qquad (2).$$

Comparing this with (A) we see that they will agree provided

$$\frac{(\alpha+2\beta)^2}{4\beta(\alpha+\beta)} = \frac{a^2}{a^2-b^2}, \quad \frac{\alpha^2(\alpha+2\beta)^2}{4\beta(\alpha+\beta)} = \frac{b^4}{a^2-b^2}.$$

This gives

$$\alpha = \frac{b^2}{a}, \quad \beta = \frac{1}{2}\frac{b}{a}(a-b) \text{ or } = -\frac{1}{2}\frac{b}{a}(a+b).$$

Hence any ellipse can be generated by the rolling of an epicycloid, or of a hypocycloid, upon its major axis, the tracing point being the centre of the fixed circle. The radius of the fixed circle is the radius of curvature at the end of the major axis; and the radius of the generating circle is half the difference, or half the sum, of this radius of curvature and of the semi-axis minor.

Cor. 1.—A parabola can be described as a roulette by the rolling of the involute of a circle upon its axis, the tracing point being the centre of the circle, and the radius of the circle being half the latus rectum.

Cor. 2.—The p-r-equation of the body centrode for a hyperbola in case above considered is

$$p^2 = \frac{a^2+b^2}{a^2}r^2 - \frac{b^2}{a^2+b^2}.$$

Cor. 3.—If the space centrode be the minor axis of the ellipse, the body centrode is one or other of the hypocycloids given by

$$\alpha = a^2/b,\ \beta = \tfrac{1}{2}a(a \pm b)/b.$$

Generation of a parabola, the space centrode being the directrix. (Fig. 60).

We have here, $4a$ denoting the latus rectum of the given parabola,

$$\frac{\mathrm{NG}}{p} = \frac{\mathrm{PM}}{\mathrm{MH}}$$

$$= \frac{\sqrt{4a(p-a)}}{2a}$$

$$\mathrm{NG} = p\sqrt{\left(\frac{p}{a} - 1\right)}.$$

Hence $$r^2 = p^2 + p^2\left(\frac{p}{a} - 1\right);$$

that is, $$ar^2 = p^3 \qquad (1).$$

This leads to the differential equation

$$d\theta = \pm \frac{a^{\frac{1}{3}}dr}{r\sqrt{\left(r^{\frac{2}{3}} - a^{\frac{2}{3}}\right)}}$$

Hence, by properly choosing the prime radius, we have

$$\theta = 3\sin^{-1}\left(\frac{a}{r}\right)^{\frac{1}{3}},$$

that is $$r = a\operatorname{cosec}^3\frac{\theta}{3} \qquad (2);$$

or, in Cartesian coordinates,

$$27a(x^2 + y^2) = (4a + y)^3 \qquad (2^1).$$

The form of this cubic is given in Fig. 61.

Cor.—If we write R in place of p, and R^2/P in place of r, in equation (1), we obtain the equation to the pedal of the generating cubic. The result is

$$\mathrm{P}^2 = a\,\mathrm{R},$$

which represents a parabola identical with the given parabola.

Hence the body centrode for a parabola, when the space centrode is the directrix, is the first negative pedal of the given parabola, with respect to its focus, the tracing point being the focus.

Generation of a straight line, the space centrode being a circle. (Fig. 62).

Let $CA = d$ be the perpendicular from the centre of the circle on the given line; and let $KCA = 0$.

Then, if a be the radius of the circle, we have

$$r = PK = a\cos\theta - d;$$
$$p = r\cos\theta.$$

Hence $r^2 = ap - dr$,

that is, $$ap = r^2 + dr \qquad (1).$$

This equation can be integrated; but we confine ourselves here to the case where $d = 0$. We then have

$$ap = r^2 \qquad (2).$$

This is the p-r-equation to a circle where diameter is a, the origin being a point on the circumference. Hence the straight line is generated by a circle whose diameter is equal to the radius of the given circle, the tracing point being on the circumference, a very familiar result (Fig. 63). It is to be observed, however, that it is only the part DE of the line that can be thus generated, and that it will be impossible to generate the rest of the line with the given circle as space centrode. Similar limitations will obviously occur in the more general case where d is not zero.

Generation of the cardioid, the space centrode being a circle passing through the cusp, whose centre lies on the axis of the curve, and whose diameter is half the length of the axis. (Fig. 64).

The polar equations to the cardioid and to the circle are

$$r = a(1 + \cos\theta) \text{ and } r = a\cos\theta$$

respectively.

Since for the cardioid $rd\theta/dr = -\cot\frac{1}{2}\theta$, if PK be the normal, we have $OPK = \frac{1}{2}\theta$.

Let OK bisect the angle POB. The $OK = KP$, and we have $2OK\cos\frac{1}{2}\theta = OP = 2a\cos^2\frac{1}{2}\theta$.

Hence $OK = a\cos\frac{1}{2}\theta$; and it follows that K lies on the circle.

If C be the centre of the circle $CKO = COK = \frac{1}{2}\theta$. It follows that CK is parallel to OP. Hence the tangent to the circle at K is perpendicular to OP.

We have therefore

$$p = \mathrm{PM} = a\cos^2\tfrac{1}{2}\theta\,;$$
$$r = \mathrm{PK} = a\cos\tfrac{1}{2}\theta.$$

Hence $$ap = r^2 \qquad (1).$$

Now (1) is the p-r-equation to a circle of diameter a, the origin being a point on the circumference. Hence the body centrode in the present case is a circle whose diameter is the same as that of the fixed circle, and the tracing point is a point on its circumference.

Cor.—We have, incidentally, a proof that the pedal of a circle with respect to a point on its circumference is the cardioid generated by a pair of circles whose diameters are each equal to the radius of the given circle.

For M is a point on the pedal of the circle OKB; and OM = MP.

Hence the locus of M is a cardioid similar to OPA, the ratio of similarity being 1 : 2.

Generation of an ellipse, the space centrode being the major auxiliary circle. (Fig. 65).

Here PK $= r$, PM $= p$. Denote PH, the perpendicular from P on the conjugate diameter by P. Then, with the usual notation, we have

$$\mathrm{KH}^2 - \mathrm{HP}^2 = \mathrm{CK}^2 - \mathrm{CP}^2,$$

that is $$(r - \mathrm{P})^2 - \mathrm{P}^2 = a^2 - \mathrm{CP}^2$$
$$= \mathrm{CD}^2 - b^2$$
$$= a^2b^2/\mathrm{P}^2 - b^2 \qquad (1).$$

Also, since $$\mathrm{KCH} \approx \mathrm{PKM},$$
$$p/r = (r - \mathrm{P})/a.$$

Hence $$\mathrm{P} = r - ap/r = (r^2 - ap)/r \qquad (2).$$

We have therefore, from (1) and (2)

$$r^2 - 2(r^2 - ap) = a^2b^2r^2/(r^2 - ap) - b^2\,;$$

that is $$2(r^2 - ap)^3 - (r^2 + b^2)(r^2 - ap)^2 + a^2b^2r^2 = 0 \qquad (3).$$

This is the p-r-equation to the body centrode. The derivation of the polar equation would obviously be difficult.

Generation of a given Curve as a Roulette when the Body Centrode is given.

Referring back to figure 56, we see that we have now the curve C given in the plane Π, and R given in the plane Π'. P being a point

fixed with respect to C, we may suppose the p-r-equation of C given with respect to P as origin say

$$f(\mathrm{P},\ \mathrm{R}) = 0 \qquad (1).$$

The problem now is, given a curve (Fig. 66) to determine the locus of a point K on the normal at P which is such that, if PM be perpendicular to the tangent of this locus at K and PM = P, PK = R, then the relation (1) shall be satisfied.

Let the co-ordinates of P and K, with reference to any pair of rectangular axes be (X, Y) and (x, y) respectively; and let the equation to (R) be

$$\phi(\mathrm{X},\ \mathrm{Y}) = 0 \qquad (2).$$

Then, since PK is normal to (R), we have

$$(\mathrm{X} - x)/\phi_{\mathrm{X}} = (\mathrm{Y} - y)/\phi_{\mathrm{Y}} \qquad (3).$$

Also, if p denote dy/dx, we have

$$\mathrm{P} = \{(\mathrm{X} - x)\, p - (\mathrm{Y} - y)\}/ \sqrt{(1 + p^2)} \qquad (4),$$

And, finally,

$$\mathrm{R}^2 = (\mathrm{X} - x)^2 + (\mathrm{Y} - y)^2 \qquad (5).$$

Between these five equations we can eliminate P, R, X, Y. The resultant is a differential equation of the first order connecting x and y, the integration of which will give the equation in Cartesian coordinates to the required space centrode.

Generation of a straight line, the body centrode being a straight line.

In this case the equations of the general theory reduce as follows:—

$$\mathrm{P} = c \qquad (1),$$

where c is the distance of the tracing point from the rolling straight line.

$$\mathrm{Y} = 0 \qquad (2),$$

if we take the generated straight line as axis of X.

$$\mathrm{X} - x = 0 \qquad (3).$$

$$\mathrm{P} = \{(\mathrm{X} - x)p - (\mathrm{Y} - y)\}/ \sqrt{(1 + p^2)} \qquad (4).$$

$$\mathrm{R}^2 = (\mathrm{X} - x)^2 + (\mathrm{Y} - y)^2 \qquad (5).$$

The result of the elimination is

$$c \sqrt{(1 + \mathrm{p}^2)} = y \qquad (6).$$

This gives

$$cdy/ \sqrt{(y^2 - c^2)} = dx.$$

Hence

$$\log(y + \sqrt{y^2 - c^2}) = x/c + \mathrm{A};$$

or, if we so choose the origin that $x=o$ when $y=c$,

$$\log\frac{y+\sqrt{y^2-c^2}}{c}=x/c.$$

This gives

$$y=\frac{c}{2}(e^{x/c}+e^{-x/c}).$$

Hence the space centrode is a catenary, whose directrix is the axis of x.

In point of fact if KM be the tangent at any point K of the catenary (Fig. 67) KP the ordinate, and PM perpendicular to KM, then we have, as is well known, PM$=c$ and KM$=$arc AK, this verifies the result we have just arrived at.

Generation of a circle, the body centrode being a straight line.

Let the distance of the tracing point from the line be b; and the radius of the circle a; then the equations required to determine the space centrode are as follows:—

$$P=b \tag{1}$$

$$X^2+Y^2=a^2 \tag{2}$$

$$(X-x)/X=(Y-y)/Y \tag{3}$$

$$P=\{(X-x)p-(Y-y)\}/\sqrt{(1+p^2)} \tag{4}$$

These lead to

$$(xp-y)\{a/\sqrt{(x^2+y^2)}-1\}=b\sqrt{(1+p^2)} \tag{6}$$

or

$$x^2d\left(\frac{y}{x}\right)\{a/\sqrt{(x^2+y^2)}-1\}=b\sqrt{(dx^2+dy^2)}.$$

Changing to polar co-ordinates this equation becomes

$$r^2d\theta\left(\frac{a}{r}-1\right)=b\sqrt{(dr^2+r^2d\theta^2)}$$

which gives

$$d\theta=\frac{\pm bdr}{r^2\sqrt{\left\{1-\frac{2a}{r}+\frac{a^2-b^2}{r^2}\right\}}}$$

This is identical with the equation obtained above for the body centrode in the generation of a circle when the space centrode is a straight line.

We have, therefore, a remarkable reciprocity between the two cases, viz.:—If C be the body centrode for generating a circle when the space centrode is a straight line at a distance b from the centre,

then C will be the space centrode for generating the same circle when the body centrode is a straight line, and the tracing point is at a distance b from A.

Generation of any Curve by Means of Identical Centrodes.

It is here understood that the body and space centrodes are to be congruent curves, and that the point of contact is always to be an identically corresponding point of the two curves.

When the position of a certain point (Q) in the plane of the given curve (R) is assumed, the problem of describing R by means of identical centrodes is determinate. In fact, either centrode is similar to the first negative pedal of R, the ratio of similarity being 1 : 2.*

This appears at once (Fig. 68) if we reflect that in the present case the space centrode C′ is the image of the body centrode C in the common tangent K. For if Q be the image of P in the tangent KM, then since P is a fixed point relative to C, Q will be a fixed point relative to C′. In other words, Q is a fixed point relative to R. Moreover, since $QM = \frac{1}{2}QP$, the locus of M is similar to R, and the locus of M is the pedal of C′, with reference to Q; or, what comes evidently to the same thing, the locus of P is the pedal of a curve similar to C′, the ratio of similarity being 2 : 1.

Hence, when the point Q is selected, the determination of the centrodes is complete.

We may, of course, regard C′ as the envelope of the perpendicular bisector of QP.

In many cases this is the simplest way of treating the problem.

We may, however, use the p-r-equation of the given curve (R).

If this equation be

$$f(p, r) = 0, \qquad (1),$$

the p-r-equation to the locus of M is

$$f(2P, 2R) = 0.$$

Hence, if (p, r) be the p-r-co-ordinates of K, we have

$$P/R = p/r, \text{ and } R = P;$$

that is $P = p^2/r$, and $R = P$.

The p-r-equation of the centrode is therefore

$$f(2p^2/r, 2p) = 0 \qquad (2).$$

---

* This theorem seems to have been first pointed out by Steiner.

Generation of a straight line by means of identical centrodes.

The equation to R in this case is

$$p = 2a \qquad (1),$$

Where $2a$ is the distance of any chosen fixed point Q from the line.

The equation to either centrode is therefore

$$2p^2/r = 2a;$$

that is,

$$p^2 = ar \qquad (2).$$

This represents a parabola, of which Q is the focus.

Hence we conclude that a straight line may be traced by any parabola rolling on an equal parabola, the tracing point being the focus, a well-known result.

Generation of a circle by means of identical centrodes (*Fig* 69).

If c be the radius of the circle, and Q the chosen position of Q at a distance d from the centre O, then the p-r-equation to the circle is

$$p = (r^2 + c^2 - d^2)/2c \qquad (1).$$

Hence the p-r-equation to the centrode is

$$2p^2/r = (4p^2 + c^2 - d^2)/2c.$$

That is

$$\frac{1}{p^2} = \frac{4}{c^2 - d^2}\left(\frac{c}{r} - 1\right) \qquad (2).$$

Comparing this with the focal p-r-equation of an ellipse, viz.—

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right),$$

we see that either centrode is an ellipse whose major axis is c, and whose minor axis is $\sqrt{c^2 - d^2}$.

It is easy to see that the major axis of the space centrode must be along QO; and since we have $\sqrt{(a^2 - b^2)} = d/2$, we see that the centre of the ellipse bisects QC.

Generation of an ellipse by identical centrodes.

Take Q at the focus.

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right)$$

$$\frac{r^2}{4p^4} = \frac{1}{b^2}\left(\frac{a}{p} - 1\right)$$

This is the polar reciprocal of a limaçon with respect to its double point.

Generation of a parabola by identical centrodes.

Take Q at the focus.

$$p^2 = ar,$$

the

$$\frac{4p^4}{r^2} = 2ap$$

$$p^3 = ar^2.$$

This is the polar reciprocal of a cardioid with respect to its cusp.

Generation of the lemniscate by identical centrodes.

Take Q at the double point.

$$a^2 p = r^3$$

$$a^2 \frac{2p^2}{r} = 8p^3$$

$$\tfrac{1}{4}a^2 = pr$$

This is an equilateral hyperbola.

Generation of an equilateral hyperbola by identical centrodes.

Take Q at the centre.

$$pr = a^2$$

$$\frac{4p^3}{r} = a^2$$

$$p^3 = \tfrac{1}{3}a^2 r.$$

This is the polar reciprocal of the lemniscate.

Trigonometrical Mnemonics.

By William Renton.

Fourth Meeting, February 11th, 1887.

GEORGE THOM, Esq., President, in the Chair.

Proof of a Geometrical Theorem.

BY W. J. MACDONALD, M.A.

The middle points of the diagonals of a complete quadrilateral are collinear.

Let ABCDEF be a complete quadrilateral (see fig. 70), and X, Y, Z, the middle points of its diagonals. To prove X, Y, Z, collinear.

Let PQR be the diagonal triangle.

Then BPDQ is a harmonic range.

$$\therefore \quad BX^2 = XPXQ$$

$$\frac{BP}{BQ} = \frac{BX+XP}{BX+XQ} = \frac{\sqrt{XP.XQ}+XP}{\sqrt{XP.XQ}+XQ} = \frac{\sqrt{XP}(\sqrt{XQ}+\sqrt{XP})}{\sqrt{XQ}(\sqrt{XP}+\sqrt{XQ})} = \frac{\sqrt{XP}}{\sqrt{XQ}}$$

Similarly, $\quad \dfrac{AP}{AR} = \dfrac{\sqrt{YP}}{\sqrt{YR}}$ and $\dfrac{QF}{FR} = \dfrac{\sqrt{ZQ}}{\sqrt{ZR}}$

Now $\quad \dfrac{PB}{BQ} \cdot \dfrac{QF}{FR} \cdot \dfrac{RA}{AP} = -1$ ($\because$ triangle PQR is cut by BF)

i.e., $\quad \dfrac{BP}{BQ} \cdot \dfrac{FQ}{FR} \cdot \dfrac{AR}{AP} = 1$

i.e., $\quad \sqrt{\dfrac{XP}{XQ} \cdot \dfrac{ZQ}{ZR} \cdot \dfrac{YR}{YP}} = 1$

$\therefore \quad \dfrac{XP}{XQ} \cdot \dfrac{ZQ}{ZR} \cdot \dfrac{YR}{YP} = 1$

$\therefore \quad \dfrac{PX}{XQ} \cdot \dfrac{QZ}{ZR} \cdot \dfrac{RY}{YP} = -1$

$\therefore$ X, Y, Z, are collinear.

On Vortices.

By C. Chree, M.A.

Helmholtz and most, if not all, subsequent writers on vortex motions have, except in obtaining the fundamental equations, confined themselves to fluid of invariable density.

In the following paper are considered some simple systems of vortices in a compressible fluid. To show that such systems are of considerable importance it is sufficient to refer to the phenomena of cyclonic storms. It may be as well, however, to state that though the vortices are here treated as compressible, the circumstances are still so different from those found in nature that the results obtained could bear only a general resemblance at most to the phenomena of storms.

For brevity, the reader is supposed to understand and have in his hands for reference, Prof. Lamb's "Motion of Fluids," the symbols employed here having the same meaning, while the equations with suffix L refer to the equations so numbered in chapter VI. of that treatise.

When the fluid is limited the velocity must everywhere be tangential to the boundary. It is shown by Lamb how straight vortices parallel to one or more plane boundaries may be treated; and it is easily seen that a vortex of any shape in presence of an infinite plane boundary requires merely the introduction on the other side of the boundary of a vortex coinciding in position with the image of the real vortex as given by a plane mirror coinciding with the infinite plane. The direction of rotation in the imaginary vortex is such that the motion it causes tangential to the plane is in the same direction as that due to the real vortex. If the fluid be compressible it is only necessary that the imaginary fluid have at every point the same density as in the corresponding real point at the instant considered. If, for instance, xy be a boundary plane, vortices parallel to oz must be supposed to extend to infinity in both directions, and the density will be the same at any two points at equal distances from xy in a perpendicular to it.

Suppose, now, an infinite straight vortex parallel to oz, and of uniform section throughout, the density being some function of z, which we shall suppose not to change sign with z, then the motion

is in two dimensions. For the strength of the vortex being throughout its length constant, we have only the functions N and P, of which the former is given by (33_L), or when the cross section is small by

$$N = -\frac{m}{\pi}\log r. \tag{1}$$

Also assuming the velocity everywhere parallel to xy, and denoting by p the pressure in the fluid, and by Z the component of the external forces parallel to oz, we have

$$Z - \frac{1}{\rho}\frac{dp}{dz} = 0.$$

Provided, then, the external forces are wholly parallel to oz, and Z is a given function of z, whether $p = k\rho$ or $= k\rho^\gamma$ where k and γ are constants, the above equation determines the relation between the density at the plane xy and at the distance z from that plane, which will exist supposing no velocity parallel to oz. If now σ be the cross section at time t, and ρ the density at any height, supposed uniform over the cross section, the equation of continuity is simply $\frac{\delta}{\delta t}(\sigma\rho) = 0$, where, as in what follows, δ denotes differentiation following the fluid; whence we see that $-\frac{1}{\rho}\frac{\delta\rho}{\delta t}$ is $=\frac{1}{\sigma}\frac{\delta\sigma}{\delta t}$, and so is independent of z.

Thus for this elementary column we get

$$P = \frac{1}{2\pi}\log r\,\frac{\delta\sigma}{\delta t}. \tag{2}$$

The velocity depending on P at a distance r from the axis of the column of varying density is radial, and is given by

$$\frac{dP}{dr} = \frac{1}{2\pi r}\frac{\delta\sigma}{\delta t}. \tag{3}$$

Thus the fluid crossing a cylinder of radius r coaxial with the column of varying density, is simply $\frac{\delta\sigma}{\delta t}$ per unit of length in unit time. It follows that as much fluid leaves as enters the space between any two such cylindrical surfaces, and so the presence of a column of varying density has no direct tendency to cause variation in the density of the surrounding fluid.

Suppose we have a single thin straight filament parallel to oz, its vorticity being of strength m and its cross section at time t being σ; then the velocity at distance r from the axis is $\frac{dP}{dr}$ along and $\frac{dN}{dr}$ perpendicular to r. Thus, taking the axis of z along the axis of the

filament, the component velocities parallel to rectangular axes of x and y are

$$u = -\frac{my}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{x}{r^2} \qquad (4),$$

$$v = \frac{mx}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{y}{r^2} \qquad (5).$$

If the vortex, though of larger cross section, be circular and ρ and ζ be both functions only of the distance from the centre the same formulae will apply.

Unless $\frac{\delta\sigma}{\delta t}$ be constant the stream lines due to such a vortex vary in position with the time; if, however, $\frac{\delta\sigma}{\delta t}$ be constant they are equiangular spirals of the type

$$r = ae^{\frac{1}{2m}\frac{\delta\sigma}{\delta t}\phi} \qquad (6),$$

denoting by ϕ an angle measured from some fixed plane through oz.

Let us next consider two thin vortices similar to the last—viz., the vortex m_1, σ_1 with its centre at the point (x_1, y_1), and the vortex m_2, σ_2 with its centre at (x_2, y_2). Then for the motion of the vortices we have

$$\frac{\delta x_2}{\delta t} = -\frac{m_1}{\pi r^2}(y_2 - y_1) + \frac{1}{2\pi}\frac{\delta\sigma_1}{\delta t}\frac{x_2 - x_1}{r^2} \qquad (7),$$

$$\frac{\delta y_2}{\delta t} = \frac{m_1}{\pi r^2}(x_2 - x_1) + \frac{1}{2\pi}\frac{\delta\sigma_1}{\delta t}\frac{y_2 - y_1}{r^2} \qquad (8),$$

$$\frac{\delta x_1}{\delta t} = -\frac{m_2}{\pi r^2}(y_1 - y_2) + \frac{1}{2\pi}\frac{\delta\sigma_2}{\delta t}\frac{x_1 - x_2}{r^2} \qquad (9),$$

$$\frac{\delta y_1}{\delta t} = \frac{m_2}{\pi r^2}(x_1 - x_2) + \frac{1}{2\pi}\frac{\delta\sigma_2}{\delta t}\frac{y_1 - y_2}{r^2} \qquad (10),$$

where $r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

These equations would also apply approximately to the case of two circular vortices of larger cross section, provided r were great compared to the radius of either.

From (7) – (10) we get

$$(x_2 - x_1)\frac{\delta}{\delta t}(x_2 - x_1) + (y_2 - y_1)\frac{\delta}{\delta t}(y_2 - y_1) = \frac{1}{2\pi}\frac{\delta}{\delta t}(\sigma_1 + \sigma_2)$$

$$\therefore \qquad r^2 = a^2 + \frac{1}{\pi}\{\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2\} \qquad (11),$$

if at the time $t = o$, $r = a$, $\sigma_1 = {}_0\sigma_1$, and $\sigma_2 = {}_0\sigma_2$.

Again, from (7) – (10) it is easy to prove

$$\frac{\delta}{\delta t}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \frac{m_1 + m_2}{\pi(x_2 - x_1)^2} \qquad (12).$$

Suppose, now, the line joining the vortices to be at time t inclined at an angle ϕ to its original direction, which is taken as axis of x; then the above equation becomes

$$\frac{1}{\sec^2\phi}\frac{\delta}{\delta t}(\tan\phi) = \frac{m_1 + m_2}{\pi r^2},$$

whence
$$\phi = \frac{m_1 + m_2}{\pi}\int_0^t \frac{dt}{a^2 + (\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2)|\pi} \qquad (13.)$$

If, then, the law of variation of σ_1 and σ_2 be given, the distance of the vortices and the direction of the line joining them follow at once from (11) and (13). Suppose, for instance,

$$\left.\begin{array}{l}\sigma_1 = {}_0\sigma_1(1 + \gamma_1 t)\\ \sigma_2 = {}_0\sigma_2(1 + \gamma_2 t)\end{array}\right\} \qquad (14),$$

where γ_1 and γ_2 are constants; then, if $\dfrac{2(m_1 + m_2)}{{}_0\sigma_1\gamma_1 + {}_0\sigma_2\gamma_2}$ be denoted by B, we get

$$r^2 = a^2\left\{1 + \frac{2(m_1 + m_2)}{\pi a^2 \mathrm{B}}t\right\} \qquad (15),$$

and
$$\phi = \frac{\mathrm{B}}{2}\log\left\{1 + 2\frac{(m_1 + m_2)}{\pi a^2 \mathrm{B}}t\right\} \qquad (16),$$

whence
$$r^2 = a^2 e^{2\phi/\mathrm{B}} \qquad (17).$$

In this case the path of one vortex relative to the other is an equiangular spiral, and the time of the n^{th} complete revolution of the line joining them is

$$t_n = \frac{\pi a^2 \mathrm{B}}{2(m_1 + m_2)}(e^{4\pi/\mathrm{B}} - 1)e^{4(n-1)\pi/\mathrm{B}} \qquad (18).$$

To find the absolute motion, let X, Y be the co-ordinates of the centre of inertia of the masses m_1 at (x_1, y_1) and m_2 at (x_2, y_2); so that

$$(m_1 + m_2)\mathrm{X} = m_1 x_1 + m_2 x_2,$$
$$(m_1 + m_2)\mathrm{Y} = m_1 y_1 + m_2 y_2.$$

Then from (7)—(10) we get

$$\left.\begin{array}{l}(m_1 + m_2)\dfrac{\delta \mathrm{X}}{\delta t} = \dfrac{1}{2\pi}\dfrac{x_2 - x_1}{r^2}\left(m_2\dfrac{\delta\sigma_1}{\delta t} - m_1\dfrac{\delta\sigma_2}{\delta t}\right)\\ (m_1 + m_2)\dfrac{\delta \mathrm{Y}}{\delta t} + \dfrac{1}{2\pi}\dfrac{y_2 - y_1}{r^2}\left(m_2\dfrac{\delta\sigma_1}{\delta t} - m_1\dfrac{\delta\sigma_2}{\delta t}\right)\end{array}\right\} \qquad (19.)$$

If $m_2 \frac{\delta\sigma_1}{\delta t} - m_1 \frac{\delta\sigma_2}{\delta t}$ vanish, which includes the more special case when σ_1 and σ_2 remain constant, we have

$$X = \text{constant}, \ Y = \text{constant}.$$

In the general case we could not integrate the expressions for X and Y; but if the relations (14) hold, we have

$$(m_1 + m_2) \frac{\delta X}{\delta t} = \frac{A}{r} \cos\phi,$$

$$(m_1 + m_2) \frac{\delta Y}{\delta t} = \frac{A}{r} \sin\phi,$$

where
$$A = \frac{m_2 \, {}_0\sigma_1\gamma_1 - m_1 \, {}_0\sigma_2\gamma_2}{2\pi}.$$

By means of (15) we may regard r as the variable instead of t, and so employing (17) get

$$(m_1 + m_2) \frac{dX}{dr} \frac{r\,\delta r}{\delta t} = A \cos\left(B \log \frac{r}{a}\right);$$

but
$$\frac{\delta r}{r\delta t} = \frac{m_1 + m_2}{\pi B},$$

$\therefore$
$$X = \int C \cos\left(B \log \frac{r}{a}\right) dr,$$

where
$$C = \frac{\pi A B}{(m_1 + m_2)^2}.$$

Integrating this by parts, and supposing when $t = o$ and $r = a$ that $X = o = Y$, we get

$$X = \frac{C}{1 + B^2}\left[r\cos\left(B\log\frac{r}{a}\right) - a + Br\sin\left(B\log\frac{r}{a}\right)\right\}\Bigg] \qquad (20).$$

Similarly we find

$$Y = \frac{C}{1 + B^2}\left[r\sin\left(B\log\frac{r}{a}\right) + B\left\{a - r\cos\left(B\log\frac{r}{a}\right)\right\}\right] \qquad (21).$$

Since r and ϕ have been already determined the motion is in this case completely investigated.

We see from (11) that in every case the vortices approach to or recede from one another according as the sum of their cross sections is diminishing or increasing, the rate of approach being independent either of the directions or magnitudes of their vorticities. Since σ_1 and σ_2 must be positive, the distance of the vortices can never be less than $\sqrt{a^2 - \frac{1}{\pi}({}_0\sigma_1 + {}_0\sigma_2)}$; and as we have practically assumed the distance a great compared to the radius of either vortex,

it follows that the distance can experience only a comparatively small diminution. In particular, in applying the results deduced on the hypothesis (14) we must limit t so that $1+\gamma_1 t$ and $1+\gamma_2 t$ are both positive.

A special interest as will be seen attaches to the case $m_2 = -m_1$. The equation (11) still holds, but in place of (13) we have $\phi=0$ and so $y_2=y_1$, the vortices being supposed initially to have the centres of their bases in the axis of x. It follows at once from the equations that $(x_2-x_1)^2=r^2=a^2+\frac{1}{\pi}(\sigma_1+\sigma_2-{}_0\sigma_1-{}_0\sigma_2)$ (22).

while $$y_2=y_1=\frac{m_1}{\pi}\int\frac{\delta t}{\sqrt{a^2+\frac{1}{\pi}(\sigma_1+\sigma_2-{}_0\sigma_1-{}_0\sigma_2)}} \qquad (23).$$

If we suppose $\frac{\delta\sigma_1}{\delta t}=\frac{\delta\sigma_2}{\delta t}$ then the equations give $\frac{\delta x_2}{\delta t}=-\frac{\delta x_1}{\delta t}$, or the vortices have equal but opposite velocities in the line joining them; and if the point half-way between them be taken as origin, we have

$$x_2=-x_1=\tfrac{1}{2}a\ \sqrt{1+2(\sigma_1-{}_0\sigma_1)/\pi a^2} \qquad (24).$$

$$y_2=y_1=\frac{m_1}{\pi a}\int\frac{\delta t}{\sqrt{1+2(\sigma_1-{}_0\sigma_1)/\pi a^2}} \qquad (25).$$

This answers to the case when a single vortex exists parallel to a boundary plane, taken as that of yz, the original distance being $\frac{a}{2}$. If, again, while $m_2=-m_1$ we have $\sigma_1+\sigma_2$ constant then $r^2=a^2$; and, taking the original position of the middle point of the line joining the vortices as origin, we get

$$x_2=\frac{a}{2}+\frac{1}{2\pi a}(\sigma_1-{}_0\sigma_1),$$

$$x_1=-\frac{a}{2}+\frac{1}{2\pi a}(\sigma_1-{}_0\sigma_1),$$

and $$y_2=y_1=\frac{m_1}{\pi a}t.$$

If, while still supposing $m_2=-m_1$ we suppose the relations (14) to hold, it is easy to prove that the vortices move along straight lines; viz. m_2 along the line $y=\frac{2m_1}{{}_0\sigma_1\gamma_1}\left(x-\frac{a}{2}\right)$ and m_1 along $y+\frac{2m_1}{{}_0\sigma_2\gamma_2}\times\left(x+\frac{a}{2}\right)=0$, the origin being at the initial position of the middle point of the line joining the vortices.

The methods here employed will also give the motion when any number of vortices exist, but the difficulty of solving the equations increases rapidly with the number of vortices. As an example suppose the planes of zx and zy to form two infinite boundaries at right angles. Let a straight vortex filament of strength m parallel to oz have at time t the cross section σ, and let (x, y) be the centre of the cross section by the plane xy. Then the planes may be supposed non-existent and the fluid infinite if we introduce three new vortices all of cross section σ. One of these has its centre at $(-x, -y)$ and is of strength m, the other two are each of strength $-m$ and have their centres at $(x, -y)$ and $(-x, y)$ respectively. The velocities of the original vortex are

$$\left.\begin{aligned} u &= \frac{m}{2\pi}\frac{x^2}{y(x^2+y^2)} + \frac{1}{4\pi}\frac{\delta\sigma}{\delta t}\frac{2x^2+y^2}{x(x^2+y^2)} \\ v &= -\frac{m}{2\pi}\frac{y^2}{x(x^2+y^2)} + \frac{1}{4\pi}\frac{\delta\sigma}{\delta t}\frac{x^2+2y^2}{y(x^2+y^2)} \end{aligned}\right\} \qquad (26).$$

To determine completely the path described would in the general case be difficult. If, however, $\frac{\delta\sigma}{\delta t} = \text{constant} = km$, and (r, ϕ) denote the polar co-ordinates answering to (x, y), it is not difficult to prove that the path of the vortex is the curve

$$\left(\frac{r}{r_0}\right)^{1+k^2}\left\{\frac{\sin 2\phi - k\cos 2\phi}{\sin 2\phi_0 - k\cos 2\phi_0}\right\}^{1+3k^2/4} = e^{k(\phi-\phi_0)/2} \qquad (27),*$$

where r_0, ϕ_0 are the original values of r, ϕ.

In any application of the preceding results to the case of vortices in the earth's atmosphere it must be observed that the vortices are here supposed to exist in a fluid limited only by infinite planes and not revolving as a whole about an axis. In the case of the earth, the vortices are at distances apart comparable with the earth's radius, and the vortex motion is directly influenced by the earth's rotation; the effect also of the rotation in modifying the atmospheric density in different latitudes is of great importance. Further, the motion considered here is only in two dimensions; while in cyclonic storms the velocity and even the direction of the wind seem often to vary at

---

* For special case of incompressible fluid see a paper by Professor Greenhill in the "Quarterly Journal of Mathematics," Vol. XV.

different altitudes above the ground, and some observers assert that in the centre of the disturbance ascending vertical currents often exist. In connection with this point it may be of interest to refer to Buys Ballot's law for cyclonic storms. In these there is a central area where the barometer is low and the wind blows round this area. According to the law in question the wind does not blow perpendicularly to the line joining the observer to the point where the barometer is lowest, but is more or less directed towards the centre of the depression. Now, in accordance with the results we have obtained, if the motion were in two dimensions this law would be true only if the section of the vortex were contracting, in which case the density would be increasing and the barometer rising at the centre of the depression. Further, the magnitude of the radial velocity would be proportional to the rate of variation with the time in the height of the barometer. If the barometer were falling throughout the area of the disturbance the direction of the wind would be on the whole outwards from the centre. Thus, supposing Buys Ballot's law well founded, we must conclude either that vertical currents do exist in the centre of cyclonic storms, or else that cyclonic depressions fill up in much less time than they take to form. It should also be noticed that the rate of fluctuation of the barometer at any one station affords no clue to the law of fluctuation of the density at the centre of the disturbance. A rapid fall, for instance, might mean merely that the storm had a rapid motion of translation, or that the density diminished rapidly in approaching the centre of the depression.

A Theorem in Algebra.

By J. L. Mackenzie.

If we have given two equations $\phi(x)=0$ and $\psi(y)=0$, it is possible to express in the form of a determinant the equation whose roots are $f(x, y)$, where f is any given rational integral function.

Let a_r, β_r, be the sums of the r^{th} powers of the roots of the given equations, and s_r of the required equation. Then

$$\begin{aligned} s_r = &\{f(a_1, b_1)\}^r + \{f(a_2, b_1)\}^r + \ldots\ldots \\ &+ \{f(a_1, b_2)\}^r + \{f(a_2, b_2)\}^r + \ldots\ldots \text{ \&c.} \end{aligned}$$

Thus to any term $\lambda x^m y^n$ in the expansion of $\{f(x, y)\}^r$, there will correspond in s_r the sum

$$\lambda a_1^m b_1^n + \lambda a_2^m b_1^n + \ldots\ldots\ldots + \lambda a_1^m b_2^n + \lambda a_2^m b_2^n + \ldots\ldots \&c.$$
$$= \lambda a_m \beta_n.$$

Hence s_r may be found by expanding $\{f(x, y)\}^r$, and substituting a_m, β_n, for x^m, y^n in every term of the expansion.

It is not difficult to extend this to many cases where f is not a rational nor an integral function, and where there are more than two equations $\phi(x)$, $\psi(y)$, &c.

When s_r is known, the required equation is obtained in the form of a determinant by eliminating $1, p_1, p_2 \ldots\ldots\ldots p_n$ between

$$u^n + p_1 u^{n-1} + p_2 u^{n-2} + \ldots\ldots\ldots + p_n = 0$$

and any n of Newton's equations

$$s_1 + p_1 = 0$$
$$s_2 + p_1 s_1 + 2p_2 = 0 \qquad \&c.$$

Taking the first n of these we get

$$\begin{vmatrix} u^n & u^{n-1} & u^{n-2} \ldots\ldots & 1 \\ s_1 & 1 & 0 \ldots\ldots & 0 \\ s_2 & s_1 & 2 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ s_n & s_{n-1} & s_{n-2} \ldots\ldots & n \end{vmatrix} = 0.$$

It is evident that the only extraneous factor in this equation is $n!$

The calculations required for finding and reducing this determinant are usually very laborious; but I have applied this method to calculate the equations of certain loci derived from two conics. Write the equations of the conics in the form

$$u_2 + u_1 + u_0 = 0,$$
$$v_2 + v_1 + v_0 = 0;$$

or, in polar co-ordinates,

$$r_1^2 - P_1 r_1 + P_2 = 0,$$
$$r_2^2 - Q_1 r_2 + Q_2 = 0,$$

where $$P_1 = -\frac{u_1 r}{u_2},\ P_2 = \frac{u_0 r_2}{u_2},$$

with similar values for Q_1 and Q_2.

Then
$$a_1 = P_1,$$
$$a_2 = P_1^2 - 2P_2,$$
$$a_3 = P_1^3 - 3P_1P_2 \qquad \&c.,$$

and similarly for β_r.

(1.) If a line through the origin cut the conic U in A, and V in B, to find the equation of the locus of a point P which divides AB in the ratio $\lambda : \mu$, $(\lambda+\mu=1)$.

Here the required equation is

$$\begin{vmatrix} r^4 & r^3 & r^2 & r & 1 \\ 2(\lambda\alpha_1+\mu\beta_1), & 1 & 0 & 0 & 0 \\ 2(\lambda^2\alpha_2+\lambda\mu\alpha_1\beta_1+\mu^2\beta_2), & s_1 & 2 & 0 & 0 \\ 2\lambda^3\alpha_3+3\lambda^2\mu\alpha_2\beta_1+3\lambda\mu^2\alpha_1\beta_2+2\mu^3\beta_3, & s_2 & s_1 & 3 & 0 \\ 2\lambda^4\alpha_4+4\lambda^3\mu\alpha_3\beta_1+6\lambda^2\mu^2\alpha_2\beta_2+4\lambda\mu^3\alpha_1\beta_3+2\lambda^4\beta_4, & s_3 & s_2 & s_1 & 4 \end{vmatrix}$$

This, when finally reduced, gives

$$\begin{aligned} &u_2^2(v_2+\mu v_1+\mu^2 v_0)^2+\lambda^2 u_1^2 v_2(v_2+\mu v_1+\mu^2 v_0) \\ &+\lambda^2 u_0 u_2(2v_2^2+2\mu v_1 v_2-2\mu^2 v_0 v_2+\mu^2 v_1^2) \\ &+\lambda u_1 u_2(\mu v_1+2v_2)(v_2+\mu v_1+\mu^2 v_0) \\ &+\lambda^3 u_0 u_1 v_2(\mu v_1+2v_2) \\ &+\lambda^4 u_0^2 v_2^2=0. \end{aligned}$$

(2.) If $OP^2=OA.\ OB$, we get by the same method,

$$\begin{aligned} &u_2^2 v_2^2(u_2 v_2-u_0 v_0)^2-u_1 v_1 u_2^2 v_2^2(u_2 v_2+u_0 v_0) \\ &+(u_2^2 v_2^2-4u_0 v_0 u_2 v_2)(u_1^2 v_0 v_2+u_0 u_2 v_1^2) \\ &+u_1^4 v_0^2 v_2^2+u_0^2 u_2^2 v_1^4=0. \end{aligned}$$

(3.) Finally, if P be the harmonic conjugate of O with respect to A and B, we get for the equation to the locus of P,

$$\begin{aligned} &(u_2 v_0-u_0 v_2)^2+(u_2 v_1+u_1 v_2)(u_1 v_0+u_0 v_1) \\ &+(4u_2 v_0+u_1 v_1+4u_0 v_2)(u_1 v_0+u_0 v_1) \\ &+4(u_1 v_0+u_0 v_1)^2+4u_0 v_0 u_1 v_1 \\ &+8u_0 v_0(u_2 v_0+u_0 v_2) \\ &+16u_0 v_0(u_1 v_0+u_0 v_1) \\ &+16u_0^2 v_0^2=0. \end{aligned}$$

A Trigonometrical Note.

By Dr J. S. Mackay.

Fifth Meeting, March 11th, 1887.

Dr George Thom, President, in the Chair.

Historical Notes on a Geometrical Theorem and its Developments. [18th century.]

By J. S. Mackay, LL.D.

The theorem is—

The distance between the circumscribed and the inscribed centres of a triangle is a mean proportional between the circumscribed radius and its excess above the inscribed diameter.

Or, in other words,

The potency of the inscribed centre of a triangle with respect to the circumscribed circle is equal to twice the rectangle under the inscribed and the circumscribed radii.

The notes are arranged as far as possible in chronological order, under the names of the various geometers who have turned their attention to the question. The first paper is given nearly in full, partly because the author of it seems to be totally unknown outside of the United Kingdom, and partly because the periodical in which the paper appeared is so rare and so difficult to get that it seems a sort of mockery to refer any one to it. Some few changes have been made on the lettering of the figures, and on the notation for certain lines.

WILLIAM CHAPPLE (1746).

The *Miscellanea Curiosa Mathematica* was begun, under the editorship of Francis Holliday, in the year 1745. It was to be published quarterly, but the fact that the first volume contained only nine numbers, and that the dedication prefixed to it is dated March 25, 1749, seems to show that it cannot have appeared at the regular intervals intended. Probably it is not rash to suppose that

the fourth number was published in 1746. That number opens with "An Essay on the properties of triangles inscribed in, and circumscribed about two given circles; by Mr William Chapple" (pp. 117-124). Before entering upon his subject Chapple remarks:—

The following enquiry into the properties of triangles inscribed in, and circumscribed about given circles, has led me to the discovery of some things relating to them, which I presume have not been hitherto taken notice of, having not met with them in any author; though an ingenious correspondent of mine, in the isle of Scilly, to whom I communicated some of the propositions hereinafter demonstrated, informs me that he had begun to consider it some years ago, but did not go through with it; however, I must acknowledge that a query of his to me, relating thereto, gave me the first hint, and induced me to pursue the subject with more attention than perhaps otherwise I might have done.

PROPOSITION 1.

The areas of all triangles, circumscribed about the same circle, are as their perimeters.

For the areas of all are equal to their perimeters multiplied into half the radius.

PROPOSITION 2.

The areas of all triangles, inscribed in the same circle, are as the products of their sides.

For the areas of all are equal to the product of their sides divided by twice the diameter of their circumscribing circle.

COROLLARY.

The areas of all triangles, inscribed and circumscribed in and about the same circles, are not only as their perimeters, but also as the products of their sides.

PROPOSITION 3.

To inscribe and circumscribe a triangle in and about two concentric circles, the radius of the greater circle must be double the radius of the lesser, and the triangle will always be equilateral.

[This is so easily established that I omit Chapple's demonstration.]

PROPOSITION 4.

To inscribe and circumscribe a triangle in and about two eccentric circles, the radius of the lesser circle must be less than half the radius of the greater circle.

The area of any triangle is equal to the product of the radius of its inscribed circle into half the perimeter; therefore, if this proposition be true, the area of any triangle doth not exceed the product of half the radius of its circumscribed circle into half the perimeter. Now a triangle whose area is equal to the product of half the radius of its circumscribed circle into half the perimeter is equilateral, by Proposition 3. So that, if this proposition be true, the greatest triangle that can be inscribed in any circle will be equilateral; which we come now to demonstrate.

[This also is so easily established that I omit Chapple's demonstration.]

PROPOSITION 5.

An infinite number of triangles may be drawn, which shall inscribe and circumscribe the same two circles; provided their diameters, with respect to each other, be limited as in the two last propositions.

For, put x, y, and z equal to the sides of any triangle circumscribed about a circle whose radius is r and inscribed in a circle whose radius is R. Then $xyz/4R =$ the area of the triangle,
and $(x+y+z)\, r/2 =$ the area.

Therefore $$xyz = 2\, Rr\, (x+y+z).$$
Hence it is plain that if the sides of the triangle were required to be found by the given diameters, the question would be capable of innumerable answers; for one of the sides at least is unlimited, and may be put equal to anything at pleasure, that doth not exceed the longest line that can be drawn within the great circle as a tangent to the lesser. And hence it appears, that if the distance of the given circles be so fixed, as that any one triangle may be inscribed and circumscribed, innumerable others may be inscribed and circumscribed in and about the same circles.

PROPOSITION 6.

The nearest distance of the peripheries of two given circles, or, which amounts to the same, the distance of their centres, in order to render it possible to inscribe and circumscribe triangles, is fixed and will be always the same.

Let two circles be so situated as that a triangle ABC can be circumscribed and inscribed; then innumerable others may be inscribed and circumscribed; but it is evident by inspection that if the lesser circle be anyhow removed from its place, the sides of those triangles cannot be tangents to it; and therefore the situation of the

two circles must be the same for them all. But if it be suspected that their situation may be altered, and yet other innumerable triangles may be circumscribed and inscribed (as the contrary is not yet made appear) let it be considered, that the lesser circle must, in its removal, move along on some one or other of the sides of the innumerable triangles that might be inscribed and circumscribed at its *first situation*, and then wherever it stops (unless it be at the same distance from the periphery of the great circle on the other side of the centre, which is in effect the same situation) the like inconveniency will follow. For it is well known that triangles circumscribed about equal circles and having one common base, will continually increase their altitude, the further the point of contact in the said base is removed from the middle thereof, till at length the two lines drawn from the extremities thereof become parallel to or diverging from each other, and the altitude of the triangle becomes infinite; consequently (when the two circles are eccentric) the vertices of *only two* of the circumscribing triangles that can be erected on the base AB can be at the periphery of the circumscribing circle; and in either of these the distance of the peripheries of the greater and lesser circles must be the same, and the triangles the same, having the same base and altitude. So that the proposition is abundantly proved; and hence we have the following

Corollary.

If one side of a triangle, inscribed in and circumscribed about two given circles, be given, the other two sides are thereby limited, and may from thence be found.

[For the method of finding them Chapple refers to the answer to a question he had proposed for solution in the *Miscellanea*. The question and answer will be given later on.]

Proposition 7.

Of the innumerable triangles that may be inscribed and circumscribed in and about two given eccentric circles, two must of course be isosceles, the vertices of which will be in the common diameter of those circles, which will cut their bases at right angles; now the content of that isosceles triangle which hath the least base and greatest altitude will be the greatest, *and that of the other the* least *of all the triangles that can be inscribed and circumscribed in the given circles.*

See figure 71.

Let S and I be the centres of the greater and less circles, and let the two isosceles triangles ABC, DEF, whose vertices are at the ends of the common diameter AD, be inscribed in the one circle and circumscribed about the other; ABC is the greatest and DEF the least of the inscribed and circumscribed triangles.

Let AD meet BC at H; then H is the middle point of BC.
Find K the middle point of AB, and join CK, meeting AH in G.
Then $AG = \frac{2}{3}AH$, $CG = \frac{2}{3}CK$, $\triangle AGC = \frac{1}{3}\triangle ABC$.

Now innumerable lines may be drawn in the semicircle AED, all of which shall be tangents to the lesser circle, and sides of inscribed and circumscribed triangles, and such lines must always be less than AC, because farther removed from S, the centre of the great circle, and will continually decrease according as their extremities are farther removed from A, so that ED will be the least of all of them. Let LN be any one of those lines, and let the circumscribing and inscribed triangle LMN be drawn; also the lines LP and NQ bisecting MN and ML, and crossing each other in O, the centre of gravity of this triangle; so that $\triangle LON = \frac{1}{3}\triangle LMN$.

Because an angle of any triangle is greater or less according as it is nearer to or farther from the centre of the inscribed circle, therefore $\angle L$ is greater than $\angle A$;
therefore LP is less than AH;
therefore $\frac{2}{3}$ LP is less than $\frac{2}{3}$ AH, that is, LO is less than AG.
Similarly NO is less than CG.
Now LN, being also less than AC, the area of $\triangle LON$ is less than the area of $\triangle AGC$;
therefore $\triangle ABC$ is greater than $\triangle LMN$.

Hence it plainly appears that the area of any circumscribing and inscribed triangle which hath one angle anywhere between A and E is less than the area of the isosceles triangle ABC, and greater than the isosceles triangle DEF. And that this takes in all the cases that can possibly happen will be evident to any one that considers that if the angle L be between A and E, the angle N will be somewhere between C and D, and the angle M somewhere between B and F; so that this takes in one-half of the periphery; and the arc AF being equal to AE, and BF equal to CE, it is but changing sides with the circle, and we have the same set of triangles again; for wherever the triangle be, if it be not isosceles, we may have another

on the other side of the centre of the great circle, which shall be equal to it.

From the three foregoing propositions we may deduce the following

COROLLARY.

If the distance of the centres of the two given circles be not so fixed as that an isosceles triangle can be inscribed and circumscribed, no triangle whatsoever can be inscribed and circumscribed.

And this suggests an easy method of finding the distance of the centres of the said given circles, and consequently the nearest distance of their peripheries; for here we may always consider the triangle as isosceles, and then the distance is found as follows.

See figure 71.

Put $\quad AH = x,\ CH = y,\ SA = R,\ IH = r;$

then $\quad DH = 2R - x$, and $CH = y = \sqrt{2Rx - x^2}$;

also $\quad AC = \sqrt{x^2 + y^2}$.

Now, sine of $\angle CAH = \dfrac{CH}{AC} = \dfrac{y}{\sqrt{x^2+y^2}}$,

and sine of $\angle CAH = \dfrac{IT}{IA} = \dfrac{r}{x-r}$;

therefore $\quad \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{r}{x-r}$.

In this equation substitute for y the value $\sqrt{2Rx - x^2}$, and complete the solution; then

$$x - R - r = \sqrt{R^2 - 2Rr} = \text{the distance of the centres;}$$

therefore $\quad x = AH = R + r + \sqrt{R^2 - 2Rr}$

and $\quad DH = R - r - \sqrt{R^2 - 2Rr}$, the nearest distance of the peripheries.

Note—That R must be equal to or greater than $2r$, or else the root is impossible, which is another proof of the fourth proposition.

The question proposed for solution by Chapple (Vol. I., p. 143) and answered by him (Vol. I., pp. 171-3) was—

Let the two diameters of the circumscribing and inscribed circles of a triangle be 94 *and* 40, *and one of its sides* 90; *quere the other two sides.*

Denoting the radius of the inscribed circle by r, that of the circumscribed by R, the given side by a, and the distance between the

middle of the given side and the point of inscribed contact on that side by e, Chapple deduces the expression

$$\sqrt{2Rr+e^2+\frac{4R^2r^2}{a^2}}+\frac{2Rr}{a}\pm e$$

for the two unknown sides, the affirmative value of e being taken for the greater, and the negative for the less.

Another question proposed by Chapple (Vol. I., p. 185) and answered by him (Vol. I., p. 196) was—

Given the hypotenuse of a right-angled triangle equal to 100, *and the nearest distance of the right angle from the periphery of the inscribed circle equal to* 8; *required the legs and area of the triangle by a simple equation, with a general theorem for questions of this nature.*

The general theorem given by Chapple at the end of his solution for finding the legs of right-angled triangles is

$$R+r\pm e=\left.\begin{matrix}\text{greater}\\\text{lesser}\end{matrix}\right\}\text{ leg.}$$

He adds a corollary: The sum of the two legs of any right-angled triangle is equal to the sum of the diameters of the inscribed and circumscribing circles.

In order to draw attention to his paper in the *Miscellanea Curiosa Mathematica*, Chapple proposed the following as the prize question in the *Ladies' Diary* for 1746:—

A gentleman has a circular garden whose diameter is 310 *yards, in which is contained a circular pond whose diameter is* 100 *yards, so situated in respect of each other that their peripheries will inscribe and circumscribe an infinite number of triangles (*i.e., *whose sides shall be tangents to the pond, and angles in the fence of the garden). He being disposed to make enclosures for different uses, and farther ornaments on his scheme begun, in order thereto applies himself to the artists of Great Britain for the dimensions of the greatest and least triangles that can be inscribed and circumscribed as aforesaid? and the nearest distance of the peripheries of the garden and pond? and for a demonstration of the truth of the pond's situation?*

ROBERT HEATH (1747).

Chapple's prize question was answered somewhat unsatisfactorily next year by Robert Heath, who was then editor of the *Ladies' Diary.* At the end of his solution he says:—This property of draw-

ing triangles about circles I discovered some years ago, as may be seen in the *Monthly Oracle;* though the proposer has greatly deserved in a long account of it from Scilly, printed in a book called the *Quarterly Miscellanea Curiosa.*

Landen, who will be mentioned later on, seems also to have answered the question. He gave in an inconvenient form the expression

$$R+r\pm\sqrt{R^2-2Rr}$$

for the altitudes of the two isosceles triangles, and a construction for finding the centre of the pond.

JOHN TURNER (1748).

Another mathematical periodical, *The Mathematician* (London, 1751), edited, according to the title-page, "By a Society of Gentlemen," according to T. S. Davies by Turner, made its first appearance in 1745. Six numbers only were published, probably annually, and in the fourth of these Turner, who had been a frequent contributor to the *Miscellanea Curiosa Mathematica,* and who had answered in it the first of the questions here mentioned as proposed by Chapple, reproposed it (p. 256) in this form:—

One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically.

From his own solution given in the next number (pp. 311-313), I extract the construction, which is as simple as possible. His demonstration is not simple enough to be worth reproducing.

See figure 72.

Upon any point S, with an interval equal to the given radius of the circumscribing circle, let the circle BACV be described, in which apply the right line BC equal to the given side of the triangle. Bisect BC with the indefinite perpendicular UV; in which take HT equal to the radius of the inscribed circle. Through T draw a parallel to BC, and upon V the point where UV intersects the circle, and with the distance VB, describe an arch cutting the parallel to BC in I; then, if through I the line VIA be drawn meeting the circle in A, and BA CA be joined, ABC will be the triangle required.

JOHN LANDEN (1755).

In his *Mathematical Lucubrations* (London, 1755) Landen devotes Part I. pp. 1–24 to "an investigation of a remarkable property of triangles described in a certain manner about a circle or an ellipsis." The investigation is given in a proposition which is subdivided into three cases with six figures, and followed by eight corollaries and the solution of twenty examples. Two cases and two figures only are reproduced here, these being all that are really required.

PROPOSITION.

See figures 74, 75.

The two circles ABM, DEF, whose centres are O and I respectively, being given in magnitude and position, let any given chord AB in the circle ABM touch the circle DEF at F; and from the extremities of that chord let two other tangents, AC BC, be drawn to the circle DEF, touching it at E and D, and intersecting each other at C: it is proposed to find the radius AS of the circle ABU circumscribing the triangle ABC.

Bisecting AB by a right line HM at right angles thereto, that line will pass through O, the centre of the given circle ABM, and also through S, the centre of the circumscribing circle ABU. Draw ID, IE, IF to the points of contact D, E, F; join CI, and draw IT parallel to AB intersecting HM in T. Join AU.

Let $\quad AO = R,\ IF = r,\ IO = d,\ AH = b,\ AS = x;$

then $\quad OH = \pm\sqrt{R^2 - b^2},\ OT = \pm\sqrt{R^2 - b^2} - r,$

$$IT = FH = \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

$$AF = AE = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}.$$

$$BF = BD = b \sim \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

Case 1. When the circle DEF falls within the triangle ABC.

$$EF = \pm\sqrt{x^2 - b^2},\ GF = x \pm \sqrt{x^2 - b^2}.$$

From the similar triangles AHU, IEC, an expression is found for CE or CD, which being added to the values of AE and BD found above, will give the values of AC and BC. Thence is obtained the value of AC + BC + AB, which being multiplied by $\frac{1}{2}r$ gives the area of triangle ABC.

Another expression for the area of triangle ABC is got by con-

sidering it as equal to half the rectangle contained by AC, BC and the sine of angle ACB, which is b/x.

If these expressions be equated to each other, and the equation solved, the value of x or AS will be found to be

$$x = \frac{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}{4r} + \frac{b^2 r}{R^2 - d^2 \mp 2r\sqrt{R_4^2 - b^2}}.$$

Landen remarks that if d^2 be equal to $R^2 - 2rR$, whatever b may be, x will be equal to R.

Case 2. When the circle DEF falls without the triangle ABC, the method of procedure is the same as before, the only changes being that GF is now $\pm\sqrt{x^2 - b^2} - x$, and that the area of triangle ABC is obtained by multiplying AC + BC − AB by $\frac{1}{2}r$. In this case the value of x or AS is found to be

$$x = \frac{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}{4r} + \frac{b^2 r}{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}.$$

Landen remarks that if in this case d^2 be equal to $R^2 + 2rR$, whatever b may be, x will be equal to R. He adds that the sign proper to $\sqrt{R^2 - b^2}$ or to $\sqrt{x^2 - b^2}$ is the upper or lower one, according as O and I, or according as S and I are on the same or contrary sides of AB.

Cor. 1. It follows from what has been said that, d being equal to $\sqrt{R^2 - 2rR}$ or $\sqrt{R^2 + 2rR}$, whatever b may be, S will fall in O, and the circle circumscribing the triangle always coincide with the given circle ABM; *a thing very remarkable!*

For this to happen and the circle DEF fall within the triangle, it is obvious R must not be less than $2r$. But, that circle falling without the triangle, the same thing may happen though R be less than $2r$, so that R be greater than $\frac{1}{4}r$.

Cor. 2. "In orthographic projections, circles having the same inclination to the plane of projection being projected into similar ellipses, and any tangent of a circle projected into an ellipsis being likewise, when projected therewith, a tangent to that ellipsis: it follows that, if, within or without any ellipsis whose transverse axis is T, a second concentric similar ellipsis be described with its transverse axis t in the same direction with T; and a third ellipsis be described, similar to the other two, with its centre anywhere in

the periphery of the second ellipsis, and having its transverse axis equal to $\frac{T^2 \sim t^2}{2T}$, and parallel to the transverse axes of the other ellipses; any tangent being drawn to this third ellipsis and continued both ways till it intersects the periphery of the first ellipsis in two points, and two other tangents being drawn to the same third ellipsis from those points of intersection, the locus where these last tangents continued intersect each other will always be in the periphery of the first ellipsis.

"The drawing of tangents in that manner will be impossible unless t be less than $3T$."

Cor. 3. Here and in the examples, by *the* escribed circle is understood the circle touching the base and the other two sides produced; its radius is denoted by ρ, and the distance of its centre from the circumscribed centre by δ.

The corollary gives, in terms of the base, the circumscribed radius and the inscribed or escribed radius, twenty-six expressions for the other two sides of the triangle, the sum, difference, and product of those sides, the perimeter of the triangle, the area, the perpendicular from the vertex to the base, the distances of the circumscribed centre from the inscribed and escribed centres, and the distances of these latter centres from the vertex. The only expression important enough to quote is

$$\delta = \sqrt{R^2 + 2\rho R}.$$

Cor. 4. This consists of twenty-two expressions for the other two sides of the triangle, the sum, difference, and product of those sides, the perimeter of the triangle, the area, the perpendicular from the vertex to the base, and the distances of the circumscribed centre from the inscribed and escribed centres in terms of the base, the inscribed or escribed radius, and the sine of the vertical angle or the cotangent of half that angle.

Cor. 5. Here the triangle is supposed to be right-angled, the base becoming the hypotenuse. Seventeen expressions in terms of the circumscribed radius and the inscribed or escribed radius are given for the other two sides, their sum, difference, product, the perimeter,

and the perpendicular from the vertex to the hypotenuse. The following are worth noting:

The product of the sides about the right angle

$$= 4rR + 2r^2 = 2r\rho = -4\rho R + 2\rho^2;$$

the perpendicular from the vertex to the hypotenuse

$$= 2r + \frac{r^2}{R} = \frac{r\rho}{R} = -2\rho + \frac{\rho^2}{R};$$

the perimeter $= 2\rho$.

The sixth, seventh, and eighth corollaries are of little interest.

The following is a statement of the examples with whose solution Landen's essay concludes:—

1. Let the radii of the circles circumscribed about, and inscribed in a triangle be 5 and 2·2 respectively, and one of its sides 8; to find the other two sides.

2. Given R, r, and the distance of the inscribed centre from the vertex, to find the base.

3. Given R, r, and the perpendicular from the vertex to the base, to find the base.

4. Given R, r, and ρ, to find the base.

5. Given R, r, and the perimeter, to find the corresponding triangle.

6. Given R, ρ, and the excess of the other two sides above the base, to find the corresponding triangle.

7. Given R and r, to find the triangle when its area is a maximum or a minimum.

8. Given the vertical angle, r, and ρ, to find the base.

9. Given the vertical angle, the sum of the sides containing it, and the perpendicular from it to the base, to find the base.

10. Given the vertical angle, the perpendicular from it to the base, and the perimeter, to find the base.

11. Given the vertical angle, the difference of the sides containing it, and the perimeter, to find the base.

12. Given the vertical angle, the product of the sides containing it, and the perimeter, to find the base.

13. Given the vertical angle, the sum, and the product of the three sides, to find the base.

14. Given the vertical angle, the distance of the inscribed centre from the vertex, and d, to find the base.

15. Given the vertical angle, d, and δ, to find the base.

16. Given the vertical angle and d, to find the triangle when the base is the greatest possible.

17. Given r, ρ, and the sum of the two sides, to find the base.

18. Given r, ρ, and the difference of the two sides, to find the base.

19. Given the three escribed radii, to find the triangle.

20. Given the inscribed radius and two escribed radii, to find the triangle.

LEONHARD EULER (1765).

In the *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tom. xi. (pp. 103–123), for the year 1765, occurs a paper* of Euler's, entitled, *Solutio facilis problematum quorumdam geometricorum difficillimorum.* He there determines the distance between the inscribed and circumscribed centres of a triangle in the following way.

See figure 73.

Let I and S be the inscribed and circumscribed centres of the triangle ABC. Draw IF, SL perpendicular to AB; join AS, and draw AM perpendicular to BC.

Denote BC, CA, AB by a, b, c, and let area $=$ A, $a+b+c=p$, $ab+ac+bc=q$, $abc=r$.

Then $$\mathrm{IF}=\frac{2\mathrm{A}}{a+b+c},\quad \mathrm{AF}=\frac{c+b-a}{2}.$$

From the similar triangles ACM, ASL,

$$\mathrm{AM} : \mathrm{CM} = \mathrm{AL} : \mathrm{SL};$$

that is, $$\frac{2\mathrm{A}}{a} : \frac{a^2+b^2-c^2}{2a} = \frac{c}{2} : \mathrm{SL};$$

therefore $$\mathrm{SL}=\frac{c(a^2+b^2-c^2)}{8\mathrm{A}}.$$

Hence $$\mathrm{AF}-\mathrm{AL}=\frac{c+b-a}{2}-\frac{c}{2}=\frac{b-a}{2};$$

and $$\mathrm{IF}-\mathrm{SL}=\frac{2\mathrm{A}}{a+b+c}-\frac{c(a^2+b^2-c^2)}{8\mathrm{A}},$$

$$=\frac{(a+b)c^3+(a^2+b^2)c^2-(a+b)(a^2+b^2)c-(a^2-b^2)^2}{8(a+b+c)\mathrm{A}}.$$

---

*An abstract is given of it in the *Proceedings* for last year. See Vol. IV., pp. 51-55.

Now $IS = (AF - AL)^2 + (IF - SL)^2$,

$$= \frac{abc}{16(a+b+c)^2A^2}\left\{\begin{array}{l} +a^5+a^4b+ab^4+abc^3-2a^3b^2-2a^2b^3 \\ +b^5+a^4c+ac^4+ab^3c-2a^3c^2-2a^2c^3 \\ +c^5+b^4c+bc^4+a^3bc-2b^3c-2^2b^2c^3 \end{array}\right\},$$

$$= \frac{abc}{16(a+b+c)A^2}\left\{\begin{array}{l} +a^4+a^2bc+2a^2b^2 \\ +b^4+ab^2c-2a^2c^2 \\ +c^4+abc^2-2b^2c^2 \end{array}\right\},$$

$$= \frac{r}{16pA^2}(p^4-4p^2q+9pr),$$

$$= \frac{r(p^3-4pq+9r)}{16A^2},$$

$$= \frac{r^2}{16A^2} - \frac{r}{p}.$$

Nicolas Fuss (1794, 1798).

The inquiry is now extended to other figures than the triangle. In the tenth volume, pp. 103–125, of the *Nova Acta Academiae Scientiarum Imperialis Petropolitanae* (Petropoli, 1797), there is a paper by Fuss, entitled, *De Quadrilateris quibus circulum tam inscribere quam circumscribere licet*, which was read on the 14th August 1794. The following is a short abstract of it. The words "encyclic" and "pericyclic" are used for the phrases "that may be inscribed in a circle," and "that may be circumscribed about a circle."

The first problem and its corollaries show how, when the four sides of an encyclic quadrilateral are given, to express by means of the sides alone, the area, the radius of the circumscribed circle, the diagonals, and the angles of the figure. When, however, the quadrilateral is pericyclic, in addition to the four sides there must be given either a point of contact or an angle to determine the rest. This is shown in the second and third problems and their corollaries. In the fourth problem and its corollaries it is proved that, of all the quadrilaterals formed by four given sides, the one whose inscribed circle is the greatest is encyclic. The fifth problem is, About a given circle to circumscribe an encyclic quadrilateral; the sixth, In a given circle to inscribe a pericyclic quadrilateral. The seventh and eighth problems show how, when the four sides, or when two sides and the contained angle, are given to construct a quadrilateral

which is both encyclic and pericyclic. The ninth and tenth problems show how, when the angles of an encyclic and pericyclic quadrilateral are given, to find the sides, the area, and the radii of the circumscribed and inscribed circles. The eleventh problem is, Given the inscribed and circumscribed radii of an encyclic and pericyclic quadrilateral, to find the distance of the centres. Two expressions are found for the square of this distance, namely,

$$R^2 + r^2 \pm r\sqrt{4^2 + rR^2},$$

where the upper sign holds for the circle which touches the sides produced of the quadrilateral, and the lower for the circle really inscribed in the quadrilateral. In the latter case Fuss remarks that $R^2 + r^2$ must be greater than $r\sqrt{4R^2 + r^2}$, that is, that R must be greater than $r\sqrt{2}$. He then adds the expression for the square of the distance between the inscribed and circumscribed centres of a triangle, namely, $\qquad R^2 - 2Rr,$
and gives the following demonstration.

See figure 72.

Let I be the centre of the circle inscribed in triangle ABC, S the circumscribed centre. From A draw AV through the centre I to meet the circumscribed circle in V. Through V draw the circumscribed diameter VU, and from I let fall IT perpendicular to VU.

Then $\qquad d^2 = \mathrm{SI}^2 = \mathrm{ST}^2 + \mathrm{IT}^2.$

But $\qquad \mathrm{ST} = \mathrm{SV} - \mathrm{TV} = R - \mathrm{TV},$

and $\qquad \mathrm{IT}^2 = \mathrm{IV}^2 - \mathrm{TV}^2;$

therefore $\qquad d^2 = R^2 - 2R \cdot \mathrm{TV} + \mathrm{IV}^2.$

Draw the chords BU, BV, CV.

Then it is evident that $\angle\,\mathrm{CHV} = \angle\,\mathrm{UBV},$

and $\qquad \angle\,\mathrm{HCV} = \angle\,\mathrm{BUV};$

therefore triangle BUV is similar to triangle HCV;

therefore $\qquad \mathrm{BV} : \mathrm{UV} = \mathrm{HV} : \mathrm{CV};$

therefore $\qquad \mathrm{BV} \cdot \mathrm{CV} = \mathrm{UV} \cdot \mathrm{HV},$

$$= 2R \cdot \mathrm{HV}.$$

Now if CI be joined, it is clear that

$$\angle\,\mathrm{CIV} = \angle\,\mathrm{IAC} + \angle\,\mathrm{ICA},$$

and $\qquad \angle\,\mathrm{ICV} = \angle\,\mathrm{ICH} + \angle\,\mathrm{VCH}.$

But $\qquad \angle\,\mathrm{ICH} = \angle\,\mathrm{ICA}$, and $\angle\,\mathrm{VCH} = \angle\,\mathrm{IAC};$

therefore $\qquad \angle\,\mathrm{CIV} = \angle\,\mathrm{ICV},$

and $$IV = CV = BV.$$
Hence $$IV^2 = BV \cdot CV,$$
$$= 2R \cdot HV.$$
Substituting this value, we have
$$d^2 = R^2 - 2R \cdot (TV - HV),$$
$$= R^2 - 2R \cdot ID,$$
$$= R^2 - 2Rr.$$

In the thirteenth volume, pp. 166–189, of the *Nova Acta* (Petropoli, 1802), Fuss returns to the inquiry regarding the inscription and circumscription of circles to certain polygons. The title of his paper is *De Polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis*; it was read on the 19th April 1798. He says that he tried various ways of resolving the same problems, as he had done in the case of the quadrilateral, for polygons of more than four sides, but without success. The fundamental formulae became so perplexed that oil and toil ("oleum et opera") were fruitlessly spent in clearing them up. Laying aside, therefore, the general problem as beset with very great difficulties, he betook himself to polygons, which may be called *symetrically irregular*, that is, which have a diameter passing through the centres of the inscribed and circumscribed circles, and dividing the polygons into two equal and similar figures. Even with this limitation, he observes, the problem is a knotty one.

The following are the enunciations of the ten problems of which the paper consists:—

1. Given the angles of a pentagon, to find its sides so that a circle of given size can be circumscribed about it.

2. Given the angles of a pentagon, to find its sides so that a circle of given size can be inscribed in it.

3. Given the sides of a pentagon and any one point of contact, to find the radius of the inscribed circle.

4. If the angles of an encyclic and pericyclic pentagon are known, to find the relation between the radii R and r.

5. To find the distance of the centres of the circles inscribed and circumscribed to a symmetrically irregular pentagon.

6. In a given circle to inscribe a symmetrically irregular pentagon which shall be pericyclic.

7. To find the distance of the centres of the circles inscribed and circumscribed to a symmetrically irregular hexagon.

8. In a given circle to inscribe a symmetrically irregular hexagon which shall be pericyclic.

9. In a given circle to inscribe a symmetrically irregular heptagon which shall be pericyclic.

10. In a given circle to inscribe a symmetrically irregular octagon which shall be pericyclic.

Geometrical Notes.

By R. E. Allardice, M.A.

1. Some points of difference between polygons with an even number and polygons with an odd number of sides.

A polygon with an odd number of sides is determined when its angles are given and it is such that a circle of given radius may be circumscribed about it; while a polygon with an even number of sides is not determined by these conditions.

To prove this it is sufficient to show that in general one polygon and only one with given angles can be inscribed in a given circle if the number of sides be odd; and that if the number of sides be even either it is impossible to inscribe any such polygon or it is possible to inscribe an infinite number.

Let ABCDE (fig. 76) be a polygon with an odd number of sides. In a circle take any point A′; make an arc A′C′ to contain an angle equal to B; an arc C′E′ to contain an angle equal to D; an arc E′B′ to contain an angle equal to A; and so on. Thus the problem is in general a possible one; but it is evident that, though no definite relation among the angles is required (except that connecting the angles of any n-gon), there are limits in which the angles must lie in order that a solution be possible. Thus when the above construction is made, the point E′ must fall between C′ and A′, the point B′, between A′ and C′, and so on.

If the angles are $A_1, A_2, A_3 \ldots\ldots A_{2n+1}$, the necessary and sufficient conditions are of the form

$$A_2 + A_4 + \ldots\ldots + A_{2n} > (n-1)\pi$$
$$A_2 + A \ + \ldots\ldots + A_{2n} + A_1 < n\pi.$$

Consider next a polygon ABCDEF (Fig. 77) with an even number of sides. Take any point A′ in a circle; make the arc A′C′ to contain an angle equal to B; an arc C′E′ to contain an angle equal to D; then if the arc E′A′ contains an angle equal to F, but not otherwise, the problem is possible. Suppose this condition to be satisfied. Take any point B′ in the arc A′C′; make the arc B′D′ to contain an angle equal to C; and the arc D′F′ to contain an angle equal to E. Then A′B′C′D′E′F′ and ABCDEF are equiangular. Hence either there is no solution or there are an infinite number.

The above condition is equivalent to $B + D + F = 2\pi$, or, in the case of a polygon with $2n$ sides,

$$A_2 + A_4 + \ldots\ldots + A_{2n} = (n - 1)\pi.$$

It may be shown that the above data involve, in both cases, the number of conditions necessary to determine an n-gon. For $n - 3$ conditions must be satisfied in order that a circle may be drawn to circumscribe a given n-gon; and one more, in all $n - 2$ conditions, in order that a *given* circle may be so drawn. These with the $n - 1$ conditions involved in the giving of the angles, give the $2n - 3$ necessary conditions. In the case of a polygon with an even number of sides, one of the conditions that a circle may be drawn to circumscribe it, is expressible in terms of the angles of the polygon alone; while for a polygon with an odd number of sides, this is not the case.

A particular case of the above is that an equi-angular polygon inscribed in a circle is necessarily regular if it have an odd number of sides, but not if it have an even number of sides.

This particular case is given in Charles Hutton's *Miscellanea Mathematica*, London, 1775, p. 271, in the form :—

"An equilateral figure inscribed in a circle is always equiangular.

"But an equiangular figure inscribed in a circle is not always equilateral, except when the number of sides is odd. If the sides be of an even number, then they may be either all equal, or else half of them will always be equal to each other, the equals being placed alternately."

[I am indebted for this reference to Dr Mackay.]

The dual of this theorem, modified slightly, as sides instead of angles are now concerned, is also true, namely—

It is, in general, possible to describe one and only one n-gon with n given lines for sides such that a circle may be inscribed in it, if n

be an odd number; but if n be an even number, either there is no such n-gon or there are an infinite number.

The case of a pentagon may be selected as the method of proof is general.

Let AB, BC, &c. (fig. 78), be the given sides; then points G, H, K, L, M, may be found in one way and in one way only, such that AG = AM, BH = BG, &c.

Let $$\mathrm{AG} = x_1,\ \mathrm{BH} = x_2,\ \&\text{c.}$$
$$\mathrm{AB} = a,\ \mathrm{BC} = b,\ \&\text{c.}$$

Then $$x_1 + x_2 = a \qquad x_2 + x_3 = b \qquad x_3 + x_4 = c$$
$$x_4 + x_5 = d \qquad x_5 + x_1 = d\,;$$

which equations give unique values for x_1, x_2, &c., namely,

$$2x_1 = a - b + c - d + e,\ \&\text{c.}$$

Suppose, now, that a pentagon could be found with these sides such that a circle could be inscribed in it (fig. 78).

Let $\mathrm{AOG} = \alpha$, $\mathrm{BOH} = \beta$, &c.

It may be shown that α, β, &c., and the radius of the circle may be determined uniquely, as follows:—

It is evident that $\alpha + \beta + \gamma + \delta + \epsilon = \pi$.

Take any line (fig. 79), and from a point P in it cut off parts PQ_1, PQ_2, &c., equal to x_1, x_2, &c.

Draw a perpendicular through P to this line; then we have to find a point R in this line, such that

$$\mathrm{PRQ}_1 + \mathrm{PRQ}_2 + \ldots\ldots = \pi.$$

It is evident that one such position and only one exists, since when R is at P each of these angles is a right angle, and when R is at an infinite distance each of them is zero, and each of the angles diminishes continuously as R moves away from P.

It is obvious that PR is the radius of the inscribed circle and that PRQ_1, PRQ_2, &c., are the angles subtended at the centre by x_1, x_2, &c.

There is thus one and only one solution of the problem.

If, however, there is an even number of sides (say six) there are the equations

$$x_1 + x_2 = a \qquad x_2 + x_3 = b \qquad x_3 + x_4 = c$$
$$x_4 + x_5 = d \qquad x_5 + x_6 = e \qquad x_6 + x_1 = f$$

which give $$a - b + c - d + e - f = o\,;$$

so that, unless this relation is satisfied, it is impossible to find any

polygon of the kind required; and if it is satisfied, there are an infinite number.

For in AB (fig. 80) take any point G; make BH = BG, CK = CH, ...FN = FM; then, on account of the above relation, AG will be equal to AN.

There are thus an infinite number of ways of dividing the sides as required; and, corresponding to each method of division, it is possible to construct a hexagon in which a circle may be inscribed, by finding the radius of the inscribed circle and the angles subtended at the centre by the segments of the sides, as in the case of a polygon of an odd number of sides.

Corollary. An equilateral polygon circumscribing a circle is necessarily regular, if it have an odd number of sides; but if it have an even number, this is not necessarily the case. In the latter case, however, alternate angles are equal.

It is obvious that an equiangular polygon circumscribing a circle is necessarily regular.

Some other cases of differences between polygons with an even and those with an odd number of sides are well known. Thus if all the sides of a polygon of the first kind are produced, a crossed polygon is formed; while if the sides of one of the second kind are produced, two polygons are formed.

There are also differences in the formation of the different crossed n-gons with the sides of a given n-gon, according as n is a prime number or not.

Again, when the middle points of the sides of a polygon are given, one and only one polygon can in general be formed, if the number of sides is odd; but either none or an infinite number, if the number of sides is even. [Dr Mackay gives me a reference for this theorem to Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire* (6th ed., pp. 10, 11); and Catalan gives a reference to Prouhet, *Nouvelles Annales*, tome III. The latter reference I have verified.]

The treatment of this latter question by means of co-ordinates is perhaps of some interest.

Let (x_1, y_1), (x_2, y_2), &c, be the co-ordinates of the vertices required; (a_1, b_1), (a_2, b_2), &c., those of the middle points.
Then

$$x_1 + x_2 = 2a_1$$
$$x_2 + x_3 = 2a_2, \text{ \&c.}$$

These equations have a determinate solution if n is odd, namely,

$$x_1 = a_1 - a_2 + a_3 - a_4 + \ldots\ldots, \text{ \&c.};$$

but if n is even there is no solution unless

$$a_1 - a_2 + a_3 - a_4 + \ldots\ldots = 0$$

or

$$a_1 + a_3 + \ldots\ldots = a_2 + a_4 + \ldots\ldots,$$

which means that the centres of gravity of the two polygons, got by taking the given points alternately, coincide. If this condition is satisfied, there are an infinite number of solutions. It may be noted that the given points may be taken in any order, with the exception that when n is even, the points taken once in odd order must always be taken in odd order.

II. If a straight line (fig. 81) meets the three diagonals AC, BD, EF, of a complete quadrilateral in the points L, M, N, and if L′, M′, N′, are the harmonic conjugates of L, M, N, with respect to A and C, B and D, E and F respectively, then L′, M′, N′, are collinear.

Since P and Q, and also M and M′, are harmonically conjugate with respect to B and D, DM′PBQM is an involution of which B and D are the double points.

Hence
$$(BMDQ) = (BM'DP),$$
that is,
$$\frac{BD.MQ}{BQ.MD} = \frac{BD.M'P}{BP.M'D};$$
$$\therefore \quad \frac{M'P}{MQ} = \frac{BP}{BQ} \cdot \frac{M'D}{MD}.$$

Similarly from the relation

$$(BM'DQ) = (BMDP),$$
$$\frac{MP}{M'Q} = \frac{BP}{BQ} \cdot \frac{MD}{M'D};$$
$$\therefore \quad \frac{MP}{MQ} = \frac{M'Q}{M'P} \cdot \frac{BP^2}{BQ^2}.$$
$$\therefore \quad \frac{MP}{MQ} \cdot \frac{LR}{LP} \cdot \frac{NQ}{NR} = \frac{M'Q}{M'P} \cdot \frac{L'P}{L'R} \cdot \frac{N'R}{N'Q} \cdot \left(\frac{BP}{BQ} \cdot \frac{CR}{CP} \cdot \frac{EQ}{ER}\right)^2$$

Now
$$\frac{MP}{MQ} \cdot \frac{LR}{LP} \cdot \frac{NQ}{NR} = 1, \text{ and } \frac{BP}{BQ} \cdot \frac{CR}{CP} \cdot \frac{EQ}{ER} = 1;$$
$$\therefore \quad \frac{M'Q}{M'P} \cdot \frac{L'P}{L'R} \cdot \frac{N'R}{N'Q} = 1;$$

and hence, L′, M′, N′, are collinear.

Cor. 1. The middle points of the diagonals are collinear.

Cor. 2. If the first corollary be proved independently, the theorem may be deduced from it by projection.

Cor. 3. The dual of this theorem is also true. It may be stated as follows:—If lines l, m, n, be drawn through any point to the three diagonal points of a complete quadrangle, and the harmonic conjugates l', m', n', of l, m, n, be taken with respect to the sides of the quadrangle that pass through these points, then l', m', n', will be concurrent.

Sixth Meeting, April 8th, 1887.

J. S. MACKAY, Esq., LL.D., in the Chair.

On the value of $\Delta^n 0^m/n^m$, when m and n are very large.

By Professor TAIT.

I had occasion, lately, to consider the following question connected with the Kinetic Theory of Gases:—

Given that there are 3.10^{20} particles in a cubic inch of air, and that each has on the average 10^{10} collisions per second; after what period of time is it even betting that any specified particle shall have collided, once at least, with each of the others?

The question obviously reduces to this:—Find m so that the terms in

$$X^m = (x_1 + x_2 + x_3 + \ldots\ldots + x_n)^m$$

which contain each of the n quantities, once at least, as a factor, shall be numerically equal to half the whole value of the expression when $x_1 = x_2 = \ldots\ldots = x_n = 1$. Thus we have

$$X^m - \Sigma(X - x_r)^m + \Sigma(X - x_r - x_s)^m - \ldots\ldots = \tfrac{1}{2}X^m$$

or

$$\Delta^n 0^m/n^m = \tfrac{1}{2}.$$

It is strange that neither Herschel, De Morgan, nor Boole, while treating differences of zero, has thought fit to state that Laplace had, long ago, given all that is necessary for the solution of such questions. The numbers $\Delta^n 0^m$ are of such importance that one would naturally expect to find in any treatise which refers to them at least a state-

ment that in the *Théorie Analytique des Probabilités* (Livre II., chap. ii., § 4) a closely approximate formula is given for their easy calculation. No doubt the process by which this formula is obtained is somewhat difficult as well as troublesome, but the existence of the formula itself should be generally known.

When it is applied to the above problem, it gives the answer in the somewhat startling form of "about 40,000 years."

P.S.—April 4, 1887.—Finding that Laplace's formula ceases to give approximate results, for very large values of m and n when these numbers are of the same order of magnitude, I applied to Prof. Cayley on the subject. He has supplied the requisite modification of the formula, and his paper has been to-night communicated to the *Royal Society of Edinburgh.*

Sur les cordes communes à une conique et à un cercle de rayon nul; Application à la théorie géométrique des foyers dans les coniques.

PAR M. MAURICE D'OCAGNE.

1. Etant donnés une conique K, dont l'équation est $K=0$, et un point P (α, β), l'équation générale des coniques qui passent par les points d'intersection de la conique K et du cercle P de rayon nul, qui a le point (α, β) pour centre, est

$$K+\lambda[(x-\alpha)^2+(y-\beta)^2]=0. \tag{1}$$

Comme les quatre points d'intersection du cercle P et de la conique K sont imaginaires, le système (1) comprend un seul couple de droites réelles Δ et Δ'. Ces droites seront dites, par analogie avec une expression proposée par Chasles,* les *conjointes du point P et de la conique K.*

Parmi les couples de cordes imaginaires communes au cercle P et à la conique K se trouvent les *droites isotropes* passant au point P

$$(x-\alpha)+i(y-\beta)=0$$
$$(x-\alpha)-i(y-\beta)=0,$$

* *Journal de Liouville,* T. III., p. 385.

c'est-à-dire, les droites qui unissent le point P aux *ombilics* I et J du plan (expression de Laguerre), points imaginaires situés sur la droite de l'infini et par où passent tous les cercles du plan.

Une première conséquence de cette remarque est que *le point P a même polaire relativement à la conique K et aux conjointes* Δ *et* Δ'. Les conjointes passent en effet, d'après ce qui précède, par les points communs à la conique K et aux droites PI et PJ.

Par conséquent, le point de rencontre des conjointes Δ et Δ' se trouve sur la polaire du point P relativement à la conique K. On peut observer, aussi, qu'en vertu d'une propriété générale des coniques passant par l'intersection d'une conique et d'un cercle, *les conjointes* Δ *et* Δ' *sont également inclinées sur les axes de la conique K.*

Une deuxième conséquence de la remarque faite plus haut est que parmi les conjointes d'un point et d'un cercle se trouve toujours la droite à l'infini du plan. L'autre conjointe, située à distance finie, est *l'axe radical du point et du cercle*, dont les propriétés sont bien connues.

On voit tout de suite que les conjointes du centre d'une ellipse

$$x^2/a^2+y^2/b^2=1 \qquad (a>b)$$

et de cette courbe sont données par

$$y=\pm ab/c\,;$$

et celles du centre d'une hyperbole

$$x^2/a^2-y^2/b^2=1,$$

par

$$x=\pm ab/c.$$

Dans le cas de l'ellipse, *les tangentes menées des extrémités du grand axe au cercle qui a pour diamètre le petit axe, coupent ce petit axe aux points par où passent les conjointes du centre, qui sont, d'ailleurs, parallèles au grand axe.* C'est la traduction de la formule

$$y=\pm ab/c.$$

2. L'importance de la considération des conjointes réside dans le théorème suivant :—

Dans la transformation par polaires réciproques relativement à un cercle, les éléments corrélatifs des foyers d'une conique sont les conjointes du centre du cercle directeur (centre de la transformation) et de la conique corrélative.

La démonstration de ce théorème est des plus simples. Les ombilics I et J étant situés sur la droite de l'infini, et les directions

isotropes OI et OJ (O est le centre de la transformation) étant rectangulaires, les ombilics I et J ont respectivement pour éléments corrélatifs les droites OJ et OI. Or, les foyers d'une conique K sont les points de rencontre réels des couples de tangentes menées à K par les ombilics I et J. Les éléments corrélatifs de ces foyers seront donc les cordes communes à la conique corrélative de K et au couple de droites OJ et OI, c'est-à-dire, les conjointes du centre O de la transformation, et de la conique corrélative de K.

A titre de corollaire immédiat de ce théorème on peut remarquer que si l'un des foyers de la conique K coïncide avec le centre O de la transformation, l'une des conjointes du point O et de la conique corrélative étant rejetée à l'infini, cette conique corrélative est un cercle, résultat bien connu dont on pénètre ainsi la raison intime.

Le théorème précédent permettra de transformer les propriétés des foyers des coniques en propriétés de conjointes, et *vice versâ.* En particulier, on pourra considérer les systèmes de coniques ayant les mêmes cordes réelles communes avec un point donné, coniques qui pourront être dites *homoconjonctives* par rapport à ce point, et toutes leurs propriétés se déduiront corrélativement des propriétés bien connues des systèmes de coniques homofocales.

Mais ici nous nous attacherons surtout—c'est là le principal objet de ce petit Mémoire—à faire voir comment le théorème précédent peut être utilisé pour déduire les propriétés focales des coniques des propriétés tout élémentaires de l'axe radical d'un cercle et d'un point.

3. Soient C un cercle, O un point, f l'axe radical de ce cercle et de ce point (droite équidistante du point O et de la polaire de ce point relativement au cercle C).* Une transformation par polaires réciproques de centre O donne comme courbe corrélative du cercle C une conique K, qui est une ellipse, une hyperbole ou une parabole, selon que le point O est à l'intérieur, à l'extérieur ou sur la circonférence du cercle C. Mais, dans tous les cas, cette conique K a pour foyers le point O et le point F corrélatif de la droite f.

4. Joignons le point O à un point M pris sur le cercle C; la tangente en M au cercle C coupe la droite f au point N; tirons ON et menous $O\mu$ parallèlement à MN.

---

* Le lecteur est prié de faire les figures.

D'après une propriété fondamentale de l'axe radical f du point O et du cercle C, on a $NO = NM$;

donc $\angle NOM = \angle NMO$, et $\angle NOM = \angle MO\mu$.*

L'élément corrélatif du point M pris sur le cercle C est une tangente m à la conique K; celui de la droite f est le foyer F de la conique K, l'autre foyer étant au point O; celui de la tangente MN au cercle C est le point de contact P de la tangente m sur la conique K; ceux des points N et μ (ce dernier situé à l'infini dans la direction MN) sont les rayons vecteurs PF et PO. La transformation de la propriété précédente montre donc que l'angle de PF avec la tangente m est égal à l'angle de m avec PO. On obtient ainsi cette propriété classique :—

La tangente en un point d'une conique est bissectrice de l'angle des rayons vecteurs qui unissent le point de contact aux foyers de la conique.

5. Les polaires d'un point M, pris sur l'axe radical de deux cercles, relativement à ces deux cercles, se coupent sur leur axe radical, car elles sont elles-mêmes les axes radicaux du cercle de centre M orthogonal aux cercles donnés et de ces cercles. Lorsque l'un des cercles donnés se réduit à un point O cette propriété devient : si la polaire, relativement à un cercle C d'un point M pris sur l'axe radical d'un point O et de ce cercle C coupe cet axe radical au point M′, l'angle MOM′ est droit. Transformant par polaires réciproques, on a ce théorème bien connu :—

Le pôle d'une droite passant par le foyer F d'une conique, relativement à cette conique est sur la perpendiculaire élevée en F à cette droite.

Ou bien :—

La perpendiculaire menée par un foyer F d'une conique au rayon vecteur d'un point M de cette conique coupe la tangente en M sur la directrice relative au foyer F.†

---

* Une transformation homographique permet de déduire de là le théorème suivant :—*Les segments d'une tangente à une conique, compris entre le point de contact de cette tangente et ses intersections avec les conjointes d'un point par rapport à la conique, sont vus de ce point sous des angles égaux.*

† On peut aussi remarquer, en observant que l'élément corrélatif du centre d'un cercle C est la directrice de la conique corrélative K qui correspond au foyer confondu avec le centre O de la transformation, que cette propriété est également corrélative de celle-ci :—

Toutes les normales à un cercle passent par le centre de ce cercle.

6. Tout cercle dont le centre M est sur la droite f et qui passe par le point O coupe orthogonalement le cercle C, c'est-à-dire qu'il passe par les points de contact des tangentes menées de M au cercle C. La transformation par polaires réciproques de cette propriété montre que si F et F′ sont les foyers d'une conique K et que MM′ soit une corde de la conique passant par le foyer F, la parabole qui a F′ pour foyer et MM′ pour directrice est tangente aux tangentes menées à la conique K par les points M et M′. En outre, si μ et μ' sont les points de contact de ces tangentes et de la parabole les angles MF′μ et M′F′μ' sont droits. Donc, d'après le théorème qui termine le No. 5, les points μ et μ' appartiennent à la directrice de la conique K relative au foyer F′. On est ainsi conduit à ce théorème qui ne nous semble pas avoir été déjà remarqué.

Une parabole qui a pour foyer un foyer F′ d'une conique K et pour directrice une droite quelconque passant par l'autre foyer F de la conique K et coupant cette conique aux points M et M′, est tangente aux tangentes à la conique K menées par les points M et M′, la corde de contact étant la directrice de la conique K relative au foyer F′.

7. Soient f et f_1 les axes radicaux d'un point O et de deux cercles C et C_1. La perpendiculaire abaissée du point de rencontre de f et f_1 sur la ligne des centres de C et C_1 est l'axe radical de ces deux cercles. La transformation par polaires réciproques, le point O étant toujours pris pour centre de la transformation, donne ce théorème qui nous semble également nouveau :—

Si deux coniques ont en commun un foyer O, ces coniques ont un pôle double réel qui est à la rencontre de la droite qui joint les autres foyers F et F_1 et de la perpendiculaire élevée en O à la droite qui joint ce foyer au point de rencontre des directrices qui lui correspondent dans les deux coniques.

8. Si t et t' sont les tangentes menées d'un point M à un cercle C, et que D soit un point pris sur la polaire du point M relativement au cercle C, la polaire d du point D passe par le point M, et les droites d et MD sont conjuguées harmoniques par rapport aux droites t et t'.

Supposons alors que le point M se trouve sur l'axe radical f du point O et du cercle C, et que nous prenions pour point D le point de la polaire de M qui se trouve sur la polaire de O ; la droite d se confond alors avec MO. Soient T et T′ les points de contact avec C

des tangentes issues de M; puisque $MT = MT' = MO$, si par le point O nous élevons à OT et à OT' les perpendiculaires ON et ON' qui coupent MT et MT' respectivement en N et en N', nous avons $TN = 2TM$, $T'N' = 2T'M$; par suite, la droite NN' est symétrique de TT' par rapport à M, et si cette droite coupe MD en E, OE est parallèle à *f*.

Cela posé, opérons une transformation par polaires réciproques de centre O. Au point M situé sur *f* correspond une droite passant par le foyer F de la conique K et coupant cette conique en deux points A et A'. Aux points N et N' correspondent les normales à la conique K en A et A', normales qui se coupent en B. Au point E correspond la parallèle à l'axe focal de la conique K, menée par le point B; si cette parallèle coupe AA' au point C, le point C est dès lors corrélatif de la droite ME; mais nous venons de voir que la droite MD' est conjuguée harmonique de MO par rapport aux tangentes MT et MT'; donc le point C est conjugué harmonique du point situé à l'infini sur AA' par rapport aux points A et A', c'est-à-dire que le point M est le milieu de AA', et nous obtenons ce théorème connu:—

Si par le point de rencontre des normales à une conique menées par les extrémités d'une corde focale, on mène une parallèle à l'axe focal de cette conique, cette droite passe par le milieu de la corde focale considérée.

9. Supposant toujours le point M situé sur l'axe radical *f* du point O et du cercle C, plaçons, maintenant, le point D à la rencontre de la polaire du point M relativement au cercle C et de la droite qui joint le centre de ce cercle au point O—c'est-a-dire, au pôle de la droite *f*. La droite *d* coincide alors avec la droite *f*, et nous voyons que MD est conjuguée harmonique de *f* par rapport aux tangentes au cercle C issues de M, ou, si ces tangentes coupent en H et en H' la parallèle à *f* menée par D, que D est le milieu de HH', et, par suite, que les angles HOD et DOH' sont égaux.

Transformons toujours par polaires réciproques:

Au pôle D de la droite *f* correspond la directrice de la conique K relative au foyer F. On a donc ce théorème connu:

Si la corde AA' d'une conique passe par le foyer F de cette conique, les droites, qui joignent les extrémités A et A' de cette corde au point de rencontre de la directrice relative au foyer F et de l'axe focal, sont également inclinées sur cet axe.

10. Considerons un cercle C de centre Ω (fig. 82), un point O et

une droite d quelconque perpendiculaire à $O\Omega$. Joignons le point O à un point P mobile sur le cercle C; menons OQ perpendiculaire à OP et QR parallèle à la tangente PT menée en P au cercle C, c'est-à-dire perpendiculaire à $P\Omega$. Nous avons

$$\angle OQR = \angle OP\Omega = \theta, \ \angle OQH = \angle PO\Omega = \phi.$$

Donc, si nous abaissons du point O sur QR la perpendiculaire OR, nous avons

$$OR = OQ\sin\theta = OH\frac{\sin\theta}{\sin\phi} = OH\frac{O\Omega}{P\Omega},$$

et comme OH, $O\Omega$ et $P\Omega$ sont constants, OR est aussi constant; par suite, la droite QR enveloppe un cercle de centre O.

Transformons par polaires réciproques. Nous avons une conique K ayant un foyer au point O. Au point P correspond une tangente t a cette conique; à la droite d, un point Δ de l'axe focal; au point Q, la perpendiculaire p abaissée de Δ sur t; à la droite QR qui joint le point Q au point situé à l'infini sur la tangente PT au cercle C, le point de rencontre de la droite p et du vecteur qui joint le foyer O au point de contact de la tangente t et de la conique K. D'ailleurs le cercle de centre O enveloppé par QR a pour corrélatif également un cercle de centre O. On a donc ce théorème connu:—

Le lieu du point de rencontre du rayon vecteur qui unit un point M mobile sur une conique à l'un des foyers de cette conique et de la perpendiculaire abaissée d'un point Δ de l'axe focal sur la tangente au point M, est un cercle ayant pour centre le foyer O.

Si on prend pour point Δ le second foyer F de la conique, on voit, en considérant le point M dans une position infiniment voisine de l'un des sommets de l'axe focal, que le rayon du cercle correspondant (cercle directeur) est égal à la longueur de cet axe. Rapprochant ce résultat du théorème obtenu au No. 4, on en déduit la propriété des rayons vecteurs dans les coniques.

11. Nous donnerons encore un exemple remarquable de la méthode que nous indiquons ici.

Supposons que le cercle C soit variable, mais ait constamment avec le point O même axe radical f. Considérons en outre un point fixe quelconque D. L'axe radical des points D et O considérés comme cercles de rayon nul est la perpendiculaire élevée à DO en son milieu. Cette droite coupe f en un point K, et l'axe radical du point D et du cercle C passe constamment par le point K.

Il en résulte que la polaire du point D relativement au cercle C passe constamment par le point D′ symétrique du point D par rapport au point K, point situé à la rencontre de DK et de la perpendiculaire élevée en O à OD.

Passons à la figure corrélative. Aux différents cercles C correspondent des coniques ayant toutes un foyer au point O et un foyer au point F, c'est-a-dire des coniques *homofocales*, et si *d* est la droite corrélative du point D, nous voyons que les pôles de cette droite par rapport aux coniques du système sont situés sur une droite *d′*, la corrélative du point D′. Le point K a pour élément corrélatif la droite qui joint le point F au symétrique du point O par rapport à la droite *d*. Cette droite coupe la droite *d* en un point M, et puisque l'angle DOD′ est droit, la droite *d* est la perpendiculaire élevée en M à la droite *d*. De là ce théorème connu :—

Les pôles d'une droite d relativement à un système de coniques homofocales sont situés sur la perpendiculaire menée à cette droite par le point où elle est coupée par la droite qui joint l'un des foyers au symétrique de l'autre par rapport à d.

Remarquant que la droite *d* est tangente au point M à une conique ayant pour foyers F et O, on peut énoncer encore ce théorème de la manière que voici.

Le lieu des pôles d'une droite d relativement à un système de coniques homofocales est la normale à celle de ces coniques qui touche la droite d, menée par le point de contact de cette conique et de cette droite.

12. Nous nous bornerons aux exemples qui précèdent pour mettre en relief la fécondité de la méthode qui consiste à déduire les propriétés focales des coniques de la théorie des axes radicaux.

Pour terminer, nous ferons observer que réciproquement toute propriété des foyers conduit corrélativement à une propriété des conjointes d'un point et d'une conique, et, plus particulièrement, de l'axe radical d'un point et d'un cercle.

Exemple.—Prenons cette propriété connue : Soit MFM′ une corde focale de l'ellipse dont le grand axe est AA′ ; si l'on prolonge MA et M′A jusqu'à leurs points de rencontre Q et Q avec la directrice qui correspond au foyer F, l'angle QFQ′ est droit.*

* Rouché et de Comberousse, *Traité de Géométrie*, T. II., 5e édit., p. 529, Ex. 916.

Appelons O le second foyer de la conique, et transformons par polaires réciproques en prenant le point O pour centre de la transformation. Nous obtenons ainsi ce théorème :—

Soient f l'axe radical d'un point O et d'un cercle C, et P le pôle de cet axe relativement au cercle C. Si les tangentes menées d'un point quelconque de f au cercle C coupent l'une des tangentes à ce cercle parallèles à f aux points I et I', et que les droites PI et PI' coupent la droite f aux points H et H', l'angle HOH' est droit.

Note on the Kinematics of a Quadrilateral.

By R. J. DALLAS.

I send a note on the following problem, a solution of which was requested of me by one of the tutors at King's College, Cambridge.

We are given a quadrilateral of four jointed bars ABCD (fig. 83). The bar CD being held fast, find the tangent to the locus of P, the intersection of DA, CB in any position ; and verify the following construction for the radius of curvature of the path of P :—

Let PQ be the third diagonal, draw through P a perpendicular to PQ meeting BA, CD in L and L' ; through L and L' draw parallels to PQ meeting AD in M and M' ; through M and M' draw perpendiculars to AD meeting the normal at P in O and O' ; then will

$$-1/\rho = 1/\mathrm{OP} + 2/\mathrm{O'P}.$$

The first part of this is easily found. The important angles in the figure have been marked thus—

$$\mathrm{CPQ} = \alpha,\ \ \mathrm{DPQ} = \beta,\ \ \mathrm{AQP} = \gamma,\ \ \mathrm{DQP} = \epsilon.$$

Making PD rock through a small angle $\delta\theta$, we have, if δs is the resulting element of arc traced out by P,

$\delta s = \mathrm{PD}\delta\theta\mathrm{cosecTPD}$, and so also, if $\phi = \angle\mathrm{PCD}$,

$\delta s = \mathrm{PC}\delta\phi\mathrm{cosecTPC}$.

Now $\delta\phi/\delta\theta = \mathrm{QD}/\mathrm{QC}$ as is well known, (see Goodeve's *Elements of Mechanism* p. 110), and thus $\dfrac{\sin\mathrm{TPC}}{\sin\mathrm{TPD}} = \dfrac{\mathrm{QD}}{\mathrm{PD}}\dfrac{\mathrm{PC}}{\mathrm{QC}}$.

Now $$\frac{\mathrm{QD}}{\mathrm{PD}} = \frac{\sin\mathrm{DPQ}}{\sin\mathrm{PQD}} \text{ and } \frac{\mathrm{PC}}{\mathrm{QC}} = \frac{\sin\mathrm{PQC}}{\sin\mathrm{QPC}},$$

therefore $$\frac{\sin\mathrm{TPC}}{\sin\mathrm{TPD}} = \frac{\sin\beta}{\sin\alpha} \text{ and } \mathrm{TPC} - \mathrm{TPD} = \beta - \alpha\,;$$

therefore $$\mathrm{TPC} = \beta \text{ and } \mathrm{TPD} = \alpha.$$

Thus TP makes an angle with DP equal to the angle QPC.

The construction in the figure is equivalent to asserting that

$$1/\rho = \sin\alpha\sin\beta(2\cot\epsilon + \cot\gamma)/\mathrm{PQ}.$$

Let PT make an angle ψ with CD, then $\psi = \pi - \theta - \alpha$ and thus $\delta\psi = -(\delta\theta + \delta\alpha)$.

Also $\delta s = \mathrm{PD}\delta\theta\operatorname{cosec}\alpha$, sensibly. We therefore must find $\delta\alpha$.

We have $\phi = \epsilon + \alpha$; therefore $\delta\alpha = \delta\phi - d\epsilon$.

To find $\delta\epsilon$ we displace PQ twice, first the end P into its new position* and then the end Q. Let δDQ be the increment of DQ.

Then $\delta\epsilon = (\delta s\sin(\alpha+\beta) - \delta\mathrm{DQ}\sin\epsilon)/\mathrm{QP}$.

To find δDQ, fix AD instead of CD, and rock DQ through an angle $\delta\theta$, then the tangent to the path of Q makes as before an angle equal to PQA with DQ (as shown by dotted line) and $\delta\mathrm{DQ} = \mathrm{DQ}\delta\theta\cot\gamma$. Thus

$$\delta\epsilon = \frac{\delta\theta}{\mathrm{PQ}}\left[\frac{\mathrm{PD}\sin(\alpha+\beta)}{\sin\alpha} - \frac{\mathrm{DQ}\sin\epsilon}{\tan\gamma}\right]$$

and thus $\qquad \delta\psi = -(\delta\theta + \delta\alpha) = -(\delta\theta + \delta\phi - \delta\epsilon)$.

From this, dividing by ds, we get

$$\frac{1}{\rho} = \frac{\sin\alpha}{\mathrm{PD}}\left[1 + \frac{\mathrm{DQ}}{\mathrm{CQ}} + \frac{\mathrm{DQ}}{\mathrm{PQ}} - \frac{\mathrm{DP}}{\mathrm{PQ}}\frac{\sin(\alpha+\beta)}{\sin\alpha}\right]$$

$$= \frac{\sin\alpha\sin\beta}{\mathrm{PQ}}\left[\frac{\mathrm{PQ}}{\mathrm{PD}\sin\beta} + \frac{\mathrm{DQ.PQ}}{\mathrm{CQ.PD}\sin\beta} + \frac{\mathrm{DQ}}{\mathrm{DP}}\frac{\cot\gamma\sin\epsilon}{\sin\beta} - \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\right]$$

$$= \frac{\sin\alpha\sin\beta}{\mathrm{PQ}}\left[\frac{\sin\theta}{\sin\epsilon\sin\beta} + \frac{\sin\phi}{\sin\epsilon\sin\alpha} + \cot\gamma - \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\right]$$

$$= \frac{\sin\alpha\sin\beta}{\mathrm{PQ}} \cdot \frac{\sin\theta\sin\alpha\sin\gamma + \sin\phi\sin\beta\sin\gamma + \cos\gamma\sin\alpha\sin\beta\sin\epsilon - \sin(\alpha+\beta)\sin\gamma\sin\epsilon}{\sin\alpha\sin\beta\sin\gamma\sin\epsilon}$$

Substituting for θ and ϕ in terms of the other angles, namely, $\theta = \beta + \epsilon$, $\phi = \alpha + \epsilon$, we readily get the result

$$-1/\rho = \sin\alpha\sin\beta(2\cot\epsilon + \cot\gamma)/\mathrm{PQ},$$

which agrees with the result already given.

---

* Evidently for this purpose we may treat the element of arc as straight.

Seventh Meeting, May 13th, 1887.

George Thom, Esq., LL.D., President, in the Chair.

Some Correspondence between Robert Simson, Professor of Mathematics in the University of Glasgow, and Matthew Stewart, Professor of Mathematics in the University of Edinburgh.

By James Taylor, M.A.

This correspondence, purchased at the sale of the Gibson-Craig collection of MSS., is now in the possession of Dr J. S. Mackay.

An Experiment in the Teaching of Geometry.

By A. Y. Fraser, M.A., F.R.S.E.

§ 1. The course of geometry here referred to was given to pupils in George Heriot's School in the session preceding that in which they should begin the usual systematic study of geometry. The chief object of the course was to furnish their minds with a number of geometrical ideas before they should meet with these ideas as treated by Euclid. Subsidiary ends were also kept in view—such as to get them to make neat and accurate figures, and to enable them to solve various practical problems of construction and measurement.

§ 2. The classes with whom the experiment was tried were three in number, containing each fifty boys. The average age of the boys at the middle of the session was 12·3, 13·2 and 13 years. The time given was one period of forty-five minutes per week.

§ 3. The school provided fifty pairs of compasses and fifty box-wood rulers ($9\frac{1}{2}''$ long, of special design, $\frac{1}{4}''$ at each end being unmarked and the inches being divided into 8ths, 10ths, 12ths, and 16ths). Each pupil had a note-book about the size of an ordinary copy book, ruled in squares $\frac{1}{3}'' \times \frac{1}{3}''$.

§ 4. The method adopted in the lessons was in the main as follows:—First the problem was worked out on the blackboard, an attempt, which was usually successful, being always made to get the pupils to discover the solution for themselves. After the ground was cleared in this way, the pupils wrote down in their books a simple statement of the construction or measurement to be effected, and this they carried out in the manner that had been finally adopted on the blackboard. A further study of the figure would bring out interesting facts, which also were noted by the pupils in their books.

To encourage accuracy, and also to make supervision of work easily possible, every construction done by the pupils was reduced sooner or later to measurement, the results obtained were given out by the pupils in answer to their names, and were noted down, and then the pupils were told what the result should have been had all their work been exact. The pupils were seated in alphabetical order, so that miraculous coincidences of results could be readily detected and investigated. With a view to the work to be done by the pupils in future sessions in the physics laboratory, they were made to enter the results of their measurements in neat tabular forms.

§ 5. Before giving any details of the propositions discussed, I wish to say that I did not attempt to draw up a systematic course beforehand; I simply made up my mind about a few groups of notions that I thought I could introduce, and the amount of time I gave to any one set of ideas was determined by the amount of useful interest I could arouse in those ideas. Whenever I found the interest likely to wane, I prepared for the introduction of a fresh discussion; the group of notions dropped being, if necessary, resumed and extended later on by way of revisal.

§ 6. I shall now give pretty full notes of two of the groups of ideas discussed (the construction and measurement of a triangle, and Euclid I. 32, with consequences) by way of specimen, and then brief notes of the rest of the work done in the twenty-five lessons gone through up to the date of the reading of this paper.

§ 7. Notes of Lessons.

Lesson 1. *Make a* Δ *with sides* 2″, $2\frac{1}{2}$″, 3″ *long* (3 *measurements.*)

(i) *having* 2″ *side as base.*

(ii) ,, $2\frac{1}{2}$″ ,,

(iii) ,, 3″ ,,

Number of measurements required to determine any rectilineal figure investigated and tabulated as follows:—

No. of Sides	3 4 5 6......n
No. of Meas[ts.]	3 5 7 9......$2n-3$

Discussion on way to find area of Δ.

Area of rectangle.

Area of Δ shown to be $= \frac{1}{2}$ circumscribing rectangle.

Rule deduced for finding Δ area, and noted.

Necessary for above to be able to draw a $\perp$ from a point on a line.

Two methods given.

Lesson 2. *Draw* $\perp$*rs from the vertices of the* Δ*s in lesson* 1, *and hence find the areas of the three* Δ*s.*

Enter results in tabular form as shown on the blackboard.

Lesson 3. *Make a quadrilateral* $ABCD$ *having given* $AB = 5''$, $BC = 3''$, $CD = 3\frac{1}{2}''$, $DA = 5\frac{1}{2}''$, $AC = 6''$ (5 *measurements*).

Find its area by finding the area of each Δ.

How to find the area of any rectilineal figure.

. . . .

Lesson 8. Show how to make an $\angle$ on one part of the page equal to an $\angle$ on another part, then give the following exercise:

Make a ΔABC *of any size and shape, and at* $\angle A$ *copy down* $\angle$*s equal to* B *and* C. *Let these be* BAD *and* DAE.

Produce CA *to any point* F.

Take a show of hands as to the coincidence or non-coincidence of AF with AE.

All but a very few will declare for coincidence or very near coincidence.

Note result (Euc. I., 32).

Note how to draw a line parallel to another.

Lesson 9. An equilateral Δ is equiangular.

Hence $\angle$ of equilateral $\Delta = \frac{1}{3}$ of $180° = 60°$.

How to make $\angle$*s* of 60°, 30°, 15°, &c. (How to bisect an $\angle$ has already been shown.)

Make an $\angle$ *of* 60° *and one of* 30° *beside it.*

Show that this is just the method (already given without explanation) of drawing a $\perp$ at end of line.

§ 8. The other chief points taken up were:

a. The construction of triangles and quadrilaterals from various data (the angles given being always 90°, 45°, 60°, 30°, and the like).

b Illustrations of Euc. I., 47.

c Discussion of figures of the same shape (their areas, &c.).

The principles of mapping and drawing to scale.

How to measure the breadth of a river without crossing it.

d Use of squared paper to find areas. A Δ described and its area found (1) by counting the whole squares included and estimating the broken ones; (2) by the ordinary method. Results compared. Area of circle found in this way.

§ 9. To the foregoing I am permitted now (March 1888) to add a few remarks.

The course above described is neither complete nor systematic. If an idea came in the regular course, good and well; if a digression was necessary we digressed. We had no text-book to follow, and no examination to prepare for. As it happened, the pupils were examined after all, and it may encourage others to know that the work was characterised as "an excellent special course."

This session the course, still subject to modification, is being repeated to four classes similar to last year's three. A problem discussed this year with great interest was the finding of the distance of an inaccessible object by actual work in the open air. The pupils got out in batches of ten (the rest of the class working at another problem). A chain was used to measure the base line, and the base angles were taken by the eye applied to a ruler laid along the paper. Precautions were taken to have the books in the proper position. Further work of this kind is to be done.

Eighth Meeting, June 10th, 1887.

GEORGE THOM, Esq., LL.D., President, in the Chair.

Note on Milner's Lamp.

By Professor TAIT.

This curious device is figured at p. 149 of De Morgan's *Budget of Paradoxes,* where it is described as a "hollow semi-cylinder, *but not with a circular curve,*" revolving on pivots. The form of the cylinder is

such that, whatever quantity of oil it may contain, it turns itself till the oil is flush with the wick, which is placed at the edge.

Refer the "curve" to polar coördinates, r and θ; the pole being on the edge, and the initial line, of length a, being drawn to the axis. Then if θ_0 correspond to the horizontal radius vector, β to any definite radius vector, it is clear that the couple due to the weight of the corresponding portion of the oil is proportional to

$$\int_{\theta_0}^{\beta} r^2 d\theta \left(a\cos\theta_0 - \frac{2}{3} r\cos(\theta - \theta_0) \right).$$

This must be balanced by the couple due to the weight of the lamp, and of the oil beyond β; and this, in turn, may be taken as proportional to

$$\cos(a + \theta_0).$$

Thus the equation is

$$a\cos\theta_0 \int_{\theta_0}^{\beta} r^2 d\theta - \frac{2}{3}\left(\cos\theta_0 \int_{\theta_0}^{\beta} r^3 \cos\theta d\theta + \sin\theta_0 \int_{\theta_0}^{\beta} r^3 \sin\theta d\theta \right)$$
$$= b^3 \cos(a + \theta_0).$$

Differentiating twice with respect to θ_0, and adding the result to the equation, we have (with θ now put for θ_0)

$$2ar^2\sin\theta - 2ar\frac{dr}{d\theta}\cos\theta + 2r^2\frac{dr}{d\theta} = 0.$$

Rejecting the factor r, and integrating, we have

$$r^2 = 2ar\cos\theta + C.$$

This denotes a *circular* cylinder, in direct contradiction to De Morgan's statement!

As it was clear that this result, involving only one arbitrary constant, could not be made to satisfy the given differential equation for all values of b, a, and β, I fancied that it could not be the complete integral. I therefore applied to Prof. Cayley, who favoured me with the following highly interesting paper. It commences with the question I asked, and finishes with an unexpectedly simple solution of Milner's problem.

It appears clear that De Morgan did not know the solution, for the curve he has sketched is obviously one of continued curvature—and he makes the guarded statement that a friend "vouched for Milner's Lamp."

On a Differential Equation and the Construction of Milner's Lamp.

By Professor CAYLEY.

What sort of an equation is

$$b^3\cos(\alpha+\theta)=a\cos\theta\int_\theta^\beta r^3 d\theta-\frac{2}{3}\left\{\cos\theta\int_\theta^\beta r^3\cos\theta d\theta+\sin\theta\int_\theta^\beta r^3\sin\theta d\theta\right\}?\quad(1)$$

Write $$X=\int_\theta^\beta r^3 d\theta,\ Y=\int_\theta^\beta r^3\cos\theta d\theta,\ Z=\int_\theta^\beta r^3\sin\theta d\theta,\qquad(2)$$

and start with the equations

$$d\theta=\frac{dX}{-r^3}=\frac{dY}{-r^3\cos\theta}=\frac{dZ}{-r^3\sin\theta}\qquad(3)$$

$$\left(\frac{d^2}{d\theta^2}+1\right)\left\{a\cos\theta.X-\frac{2}{3}(Y\cos\theta+Z\sin\theta)\right\}=0.\qquad(4)$$

This last gives $(r-a\cos\theta)dr+ar\sin\theta.d\theta=0,$ (5)

and the system thus is

$$d\theta=\frac{dX}{-r^3}=\frac{dY}{-r^3\cos\theta}=\frac{dZ}{-r^3\sin\theta}=\frac{(r-a\cos\theta)dr}{-ar\sin\theta},\qquad(6)$$

viz., this is a system of ordinary differential equations between the five variables θ, r, X, Y, Z: the system can therefore be integrated with 4 arbitrary constants, and these may be so determined that for the value β of θ, X, Y, Z shall be each $=0$; and r shall have the value r_0.

But this being so, from the assumed equations (3) and (4) we have

$$X=\int_\theta^\beta r^3 d\theta,\ Y=\int_\theta^\beta r^3\cos\theta d\theta,\ Z=\int_\theta^\beta r^3\sin\theta d\theta$$

and further (by integration of 4)

$$L\cos\theta+M\sin\theta=a\cos\theta.X-\frac{2}{3}(Y\cos\theta+Z\sin\theta).$$

Where L and M denote properly determined constants: viz., the conclusion is that r, X, Y, Z admit of being determined as functions of θ and of an arbitrary constant r_0, in such wise that

$$a\cos\theta.X-\frac{2}{3}(Y\cos\theta+Z\sin\theta)$$

shall be a function of θ, of the proper form $L\cos\theta+M\sin\theta$, but not so that it shall be the precise function $b^3\cos(\alpha+\theta)$. To make it have

this value we must have $L = b^3\cos a$, $M = -b^3\sin a$ (where L, M are given functions of a, β, r_0), *i.e.*, we must have *two* given relations between a, b, α, β, r_0: or treating r_0 as a disposable constant we must have *one* given relation between a, b, α, β.

The equation $d\theta = \dfrac{r - a\cos\theta}{-ar\sin\theta}dr$ gives $r^2 - 2ar\cos\theta = C$. ($C = r_0^2 - 2ar_0 \cos\beta$). There would be considerable difficulty in working the question out with r_0 arbitrary, but we may do it easily enough for the particular value $r_0 = 0$ or $r_0 = 2a\cos\beta$, giving $C = 0$ and $\therefore r = 2a\cos\theta$: and we ought in this case to be able to satisfy the given equation not in general but with *two* determinate relations between the constants a, b, α, β.

We have

$$\int \cos^2\theta d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$\int \cos^4\theta d\theta = \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta$$

$$\int \cos^3\theta \sin\theta d\theta = -\frac{1}{4}\cos^4\theta$$

And thence

$$a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta)$$

$$= 4a^3\cos\theta\left\{\frac{1}{2}(\beta - \theta) + \frac{1}{4}(\sin 2\beta - \sin 2\theta)\right\}$$

$$- \frac{16}{3}a^3\cos\theta\left\{\frac{3}{8}(\beta - \theta) + \frac{1}{4}(\sin 2\beta - \sin 2\theta) + \frac{1}{32}(\sin 4\beta - \sin 4\theta)\right\}$$

$$- \frac{16}{3}a^3(\sin\theta\left\{ \qquad\qquad - \frac{1}{4}(\cos^4\beta - \cos^4\theta)\right\}$$

$$= -\frac{1}{3}a^3\cos\theta(\sin 2\beta - \sin 2\theta)$$

$$- \frac{1}{6}a^3\cos\theta(\sin 4\beta - \sin 4\theta)$$

$$+ \frac{4}{3}a^3\sin\theta(\cos^4\beta - \cos^4\theta)$$

Where the terms containing β are readily reduced to $\frac{4}{3}a^3\cos^3\beta \sin(\theta - \beta)$; hence also the terms without β disappear of themselves: and we have

$$a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta) = \frac{4}{3}a^3\cos^3\beta.\sin(\theta - \beta),$$

which may be put $= b^3\cos(\theta + \alpha)$:

viz., this will be so if we have the *two* relations

$$a = \frac{\pi}{2} - \beta\ ;\ \text{and}\ b^2 = -\frac{4}{3}a^2\cos^2\beta.$$

I make (see fig. 84) Milner's lamp, with a circular section, β arbitrary, but a segment AM ($\angle$ SAM $= \beta$) made solid. G in the line SG at right angles to AM is the C.G. of the lamp, and G′ the C.G. of the oil.

And this seems to be the *only* form—for the pole of r must, it seems to me, be *on* the bounding circle—viz., in the equation $r^2 - 2ar\cos\theta = \text{C}$, we must have $\text{C} = 0$.

An Exercise on Logarithmic Tables.

By Professor TAIT.

In reducing some experiments, I noticed that the logarithm of 237 is about 2.37 … . Hence it occurred to me to find in what cases the figures of a number and of its common logarithm are identical :—*i.e.*, to solve the equation

$$\log_{10}x = x/10^m,$$

where m is any positive integer.

It is easy to see that, in all cases, there are two solutions ; one greater than, the other less than, ϵ. This follows at once from the position of the maximum ordinate of the curve

$$y = (\log x)/x.$$

The smaller root is, for

$$m = 1,\ x = 1{\cdot}371288\ \ \dots\ \ \dots$$
$$m = 2,\ x = 1{\cdot}023855\ \ \dots\ \ \dots$$

For higher values of m, it differs but little from 1, and the excess may be calculated approximately from

$$y - y^2/2 + \ \ \dots\ \ = (1+y)\log_\epsilon 10/10^m.$$

Ultimately, therefore, the value of the smaller root is

$$1.00\ \ \dots\ \ \dots\ \ 0230258\ \ \dots\ \ \dots$$

where the number of cyphers following the decimal point is $m-1$.

The greater root must have $m+p$ places of figures before the decimal point ; p being unit till $m=9$, thenceforth 2 till $m=98$, 3 till $m=997$, &c. Thus, for example, if $m>8<98$ we may assume

$$x = (m+1)10^m + y$$

so that

$$\log_{10}\frac{m+1}{10}+\log_{10}\left(1+\frac{y}{(m+1)10^m}\right)=\frac{y}{10^m}$$

which is easily solved by successive approximations.

But it is simpler, and forms a capital exercise, to find, say to six places, the greater root, by mere inspection of a good Table of Logarithms.

Thus we find, for instance,

m	x
17	$182{,}615.10^{13}$
18	$192{,}852.10^{14}$
96	$979{,}911.10^{92}$
97	$989{,}956.10^{93}$

Geometrical Proof of the Tangency of the Inscribed and Nine-Point Circles.

By William Harvey, B.A.

S (fig. 85) is the circumscribed centre, and O the orthocentre of the triangle ABC; AX the perpendicular from A on BC, and P the middle point of BC.

SP produced bisects the arc BC in V, and I, the centre of the inscribed circle, lies on AV, and is so situated that $AI.IV=2Rr$. (*See Note*). Also the angle $XAV=$ angle $AVS=$ angle SAV.

N, the centre of the nine-point circle, bisects the distance OS, and the circumference passes through P, X and L, the middle point of AO. Hence N bisects both LP and OS, and

$$SP=OL=AL;$$

therefore LP is parallel to AS.

NHM is a radius of the nine-point circle, bisecting the chord XP in H, and the arc XP in M; ID is a radius of the inscribed circle.

Since the chord XMP is bisected at M,

$$\text{the angle } XPM=\frac{1}{2}\text{angle } XLP,$$
$$=\frac{1}{2}\text{ angle } XAS,$$
$$=\text{angle } XAV \text{ or } AVS.$$

Hence, if through I we draw a straight line (not shown in the figure) parallel to BC to meet AX and SV, the segments of this line are respectively equal to XD and PD, and we have by similar triangles

$$XD : IA = HM : MP;$$

$$PD : IV = HM : MP;$$

and therefore

$$XD.PD : IA.IV = HM^2 : MP^2.$$

But $\quad IA.IV = 2Rr$, and $MP^2 = R.HM$.

Hence $\quad XD.PD = 2r.HM$.

But $\quad HP^2 - HD^2 = XD.PD = 2r.HM$.

Now

$$\begin{aligned} IN^2 &= (NH - r)^2 + HD^2, \\ &= (NH - r)^2 + HP^2 - 2r.HM, \\ &= NH^2 + HP^2 - 2r.(NH + HM) + r^2, \\ &= \frac{R^2}{4} - rR + r^2; \end{aligned}$$

Or

$$IN = \frac{R}{2} - r.$$

Whence the tangency of the circles is evident.

Note.—The following is a simple proof of the theorem $AI.IV = 2Rr$, assumed in the foregoing proof :—

Draw IE perpendicular to AC, and ST perpendicular to VC.

Since angle AVC = angle B = 2 angle IBC, I lies on the circumference of a circle whose centre is V and radius VB or VC.

Hence $\quad IV = VC$.

Again, the triangles AIE, SVT are clearly similar;

$$\therefore\ AI : IE = SV : VT.$$

But

$$VT = \frac{1}{2}VC = \frac{1}{2}IV;$$

$$\therefore\ AI.IV = 2Rr.$$

Fig. 1

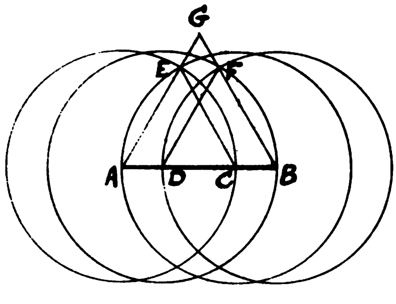


Fig. 2

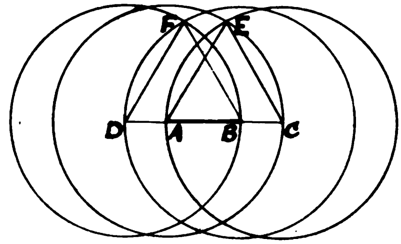


Fig. 3

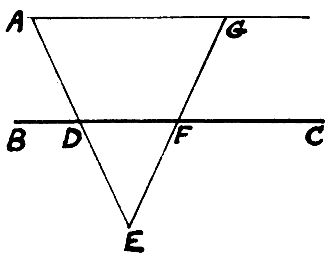


Fig. 4

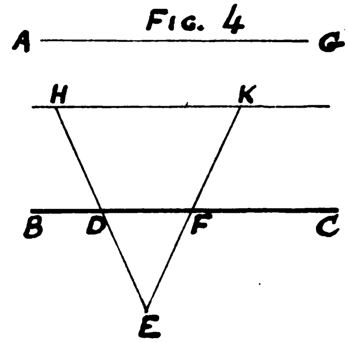


Fig. 5

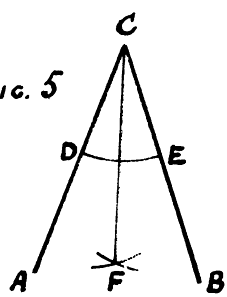


Fig. 6

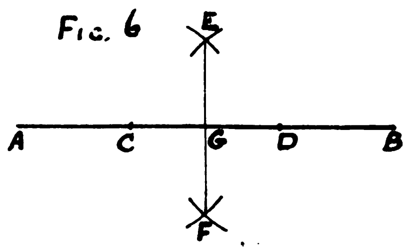


Fig. 7

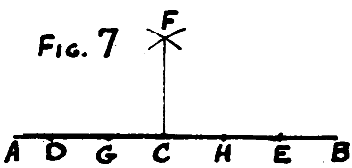


Fig. 8

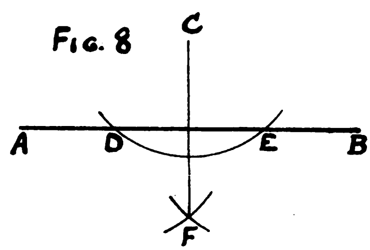


Fig. 9

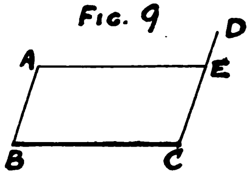


Fig. 10

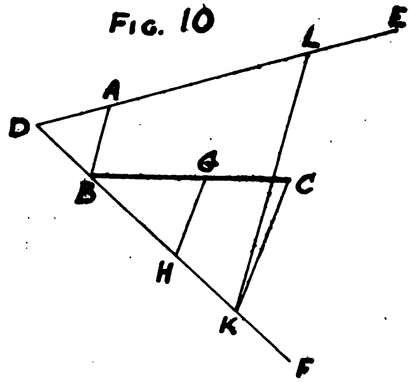


Fig. 11

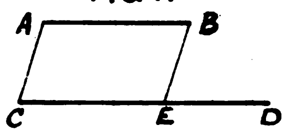


Fig. 12

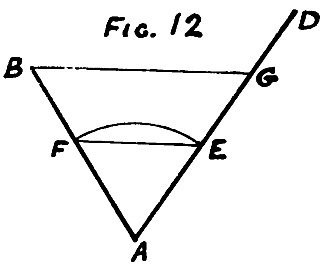


Fig. 13

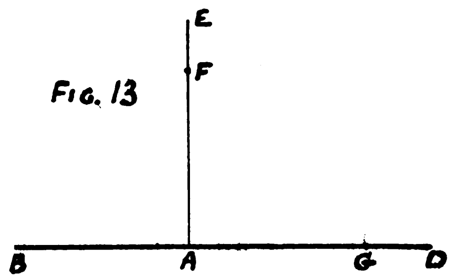


Fig. 14

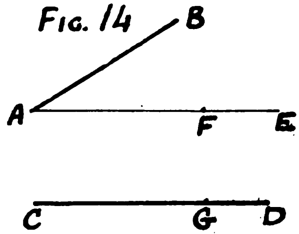


Fig. 15

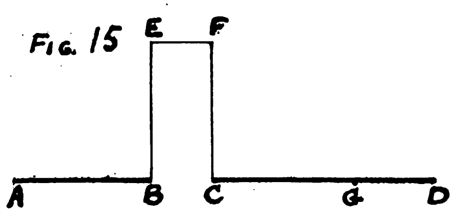


Fig. 16

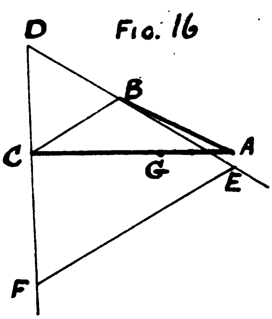
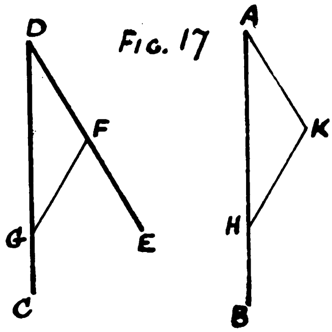


Fig. 17



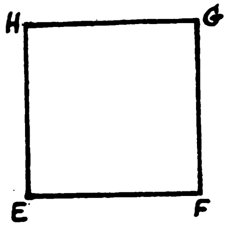


Fig. 18

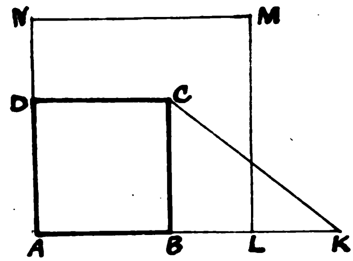


Fig. 19

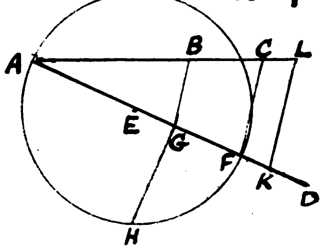


Fig. 20

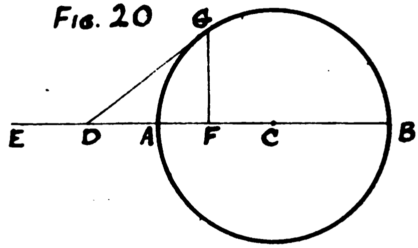


Fig. 21

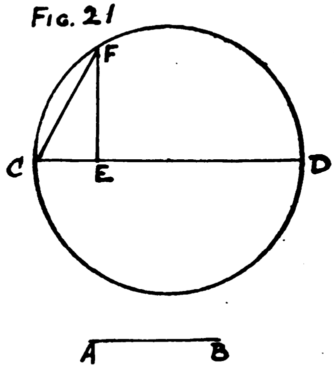


Fig. 23

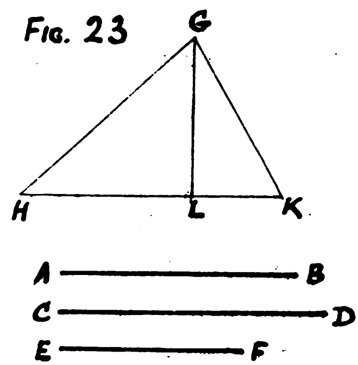


Fig. 22

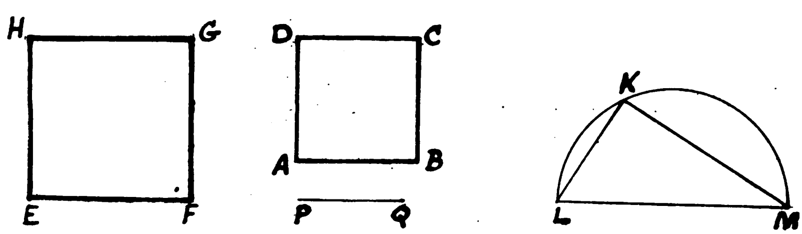


Fig. 24

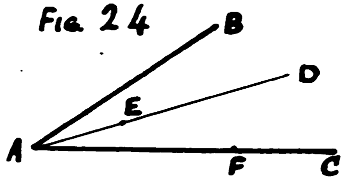


Fig. 25

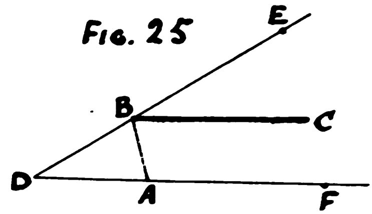


Fig. 26

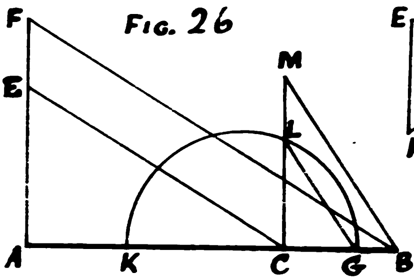


Fig. 27

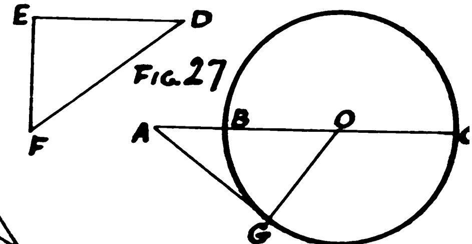


Fig. 28

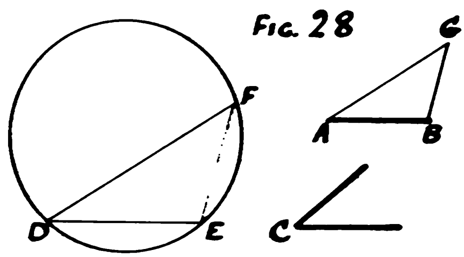


Fig. 29

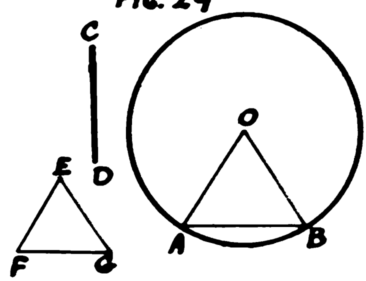


Fig. 30

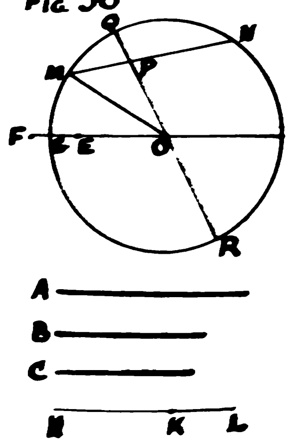


Fig. 31

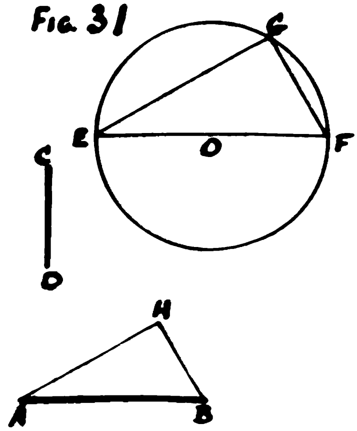


Fig. 47

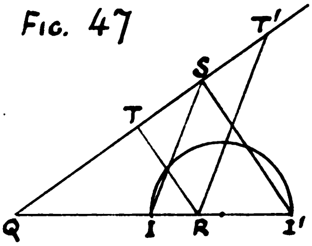


Fig. 48

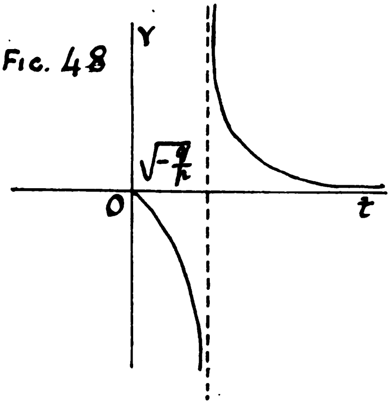


Fig. 49

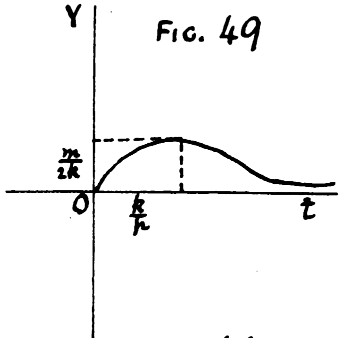


Fig. 50

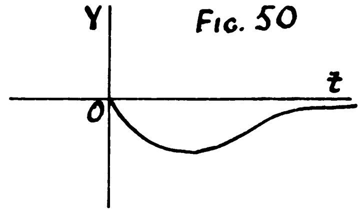


Fig. 51

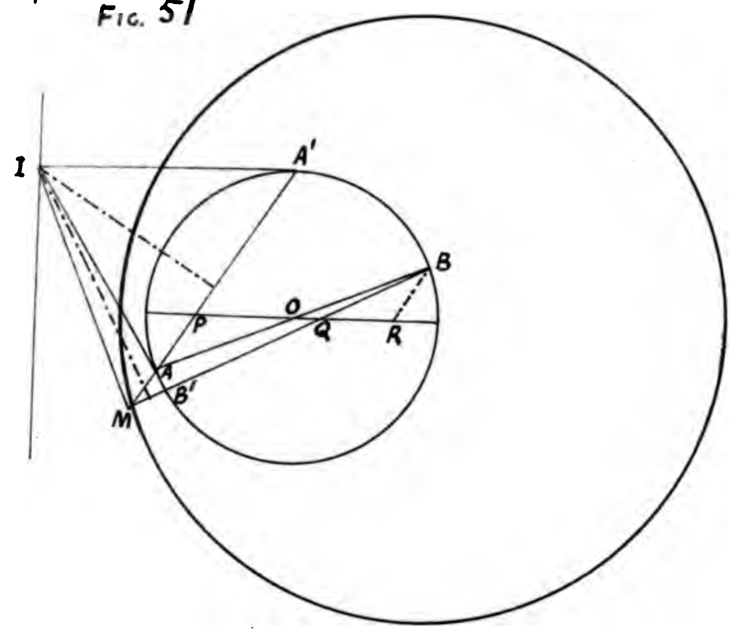
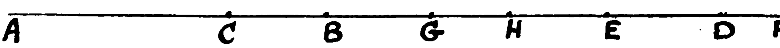


FIG. 52



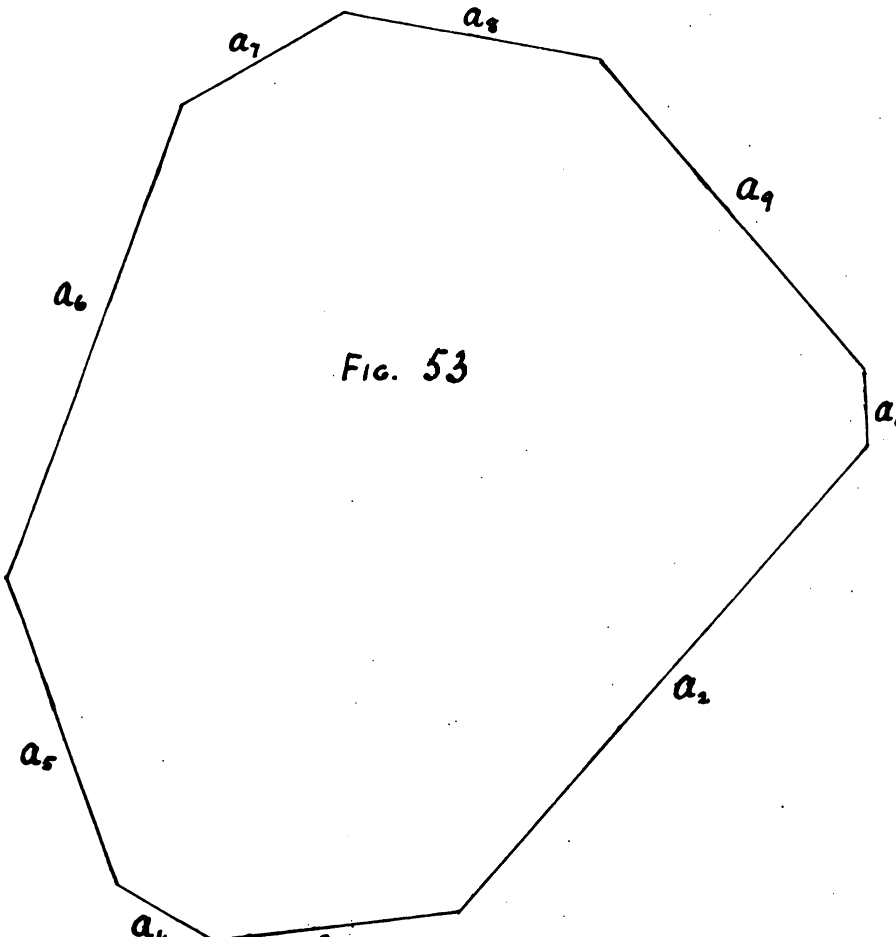


FIG. 53

Fig. 18

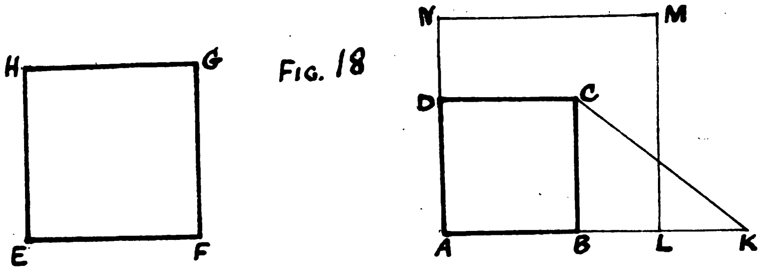


Fig. 19

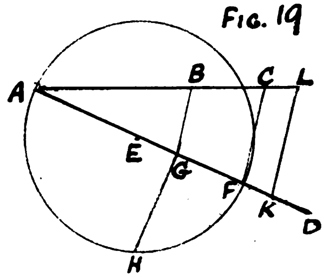


Fig. 20

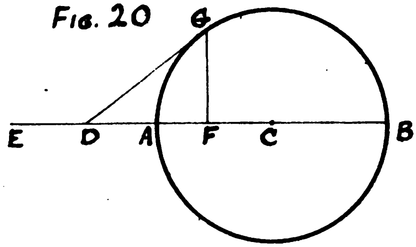


Fig. 21

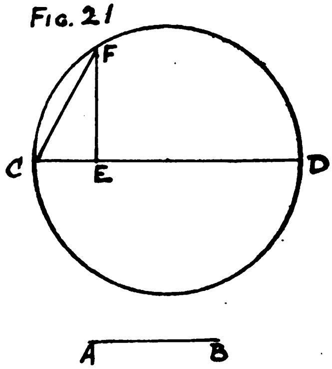


Fig. 23

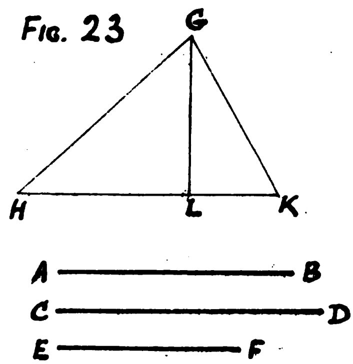


Fig. 22

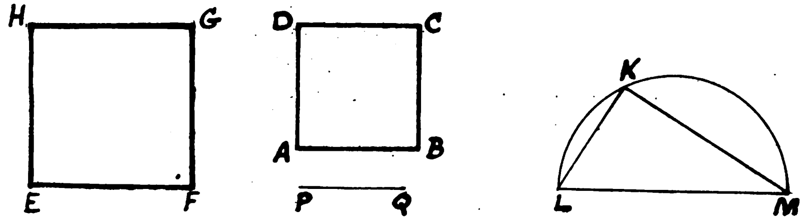


Fig. 24

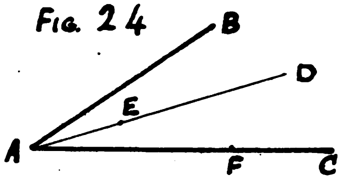


Fig. 25

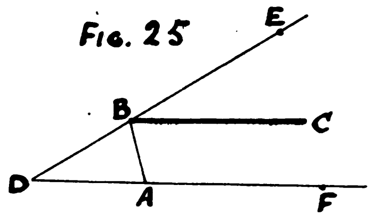


Fig. 26

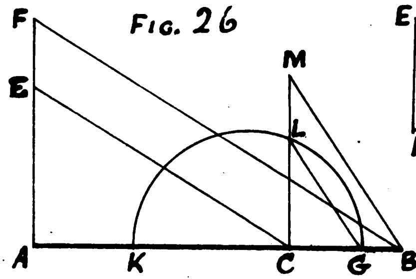


Fig. 27

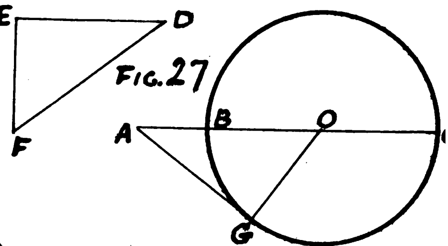


Fig. 28

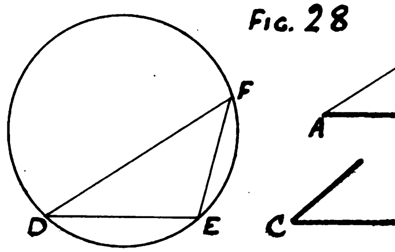


Fig. 29

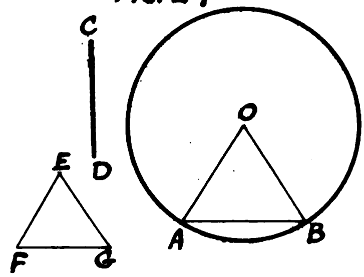


Fig. 30

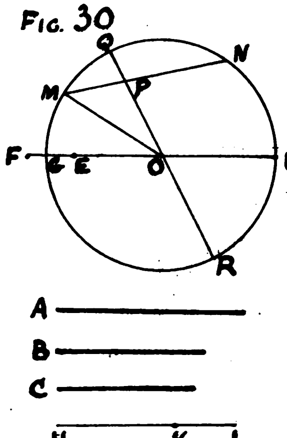


Fig. 31

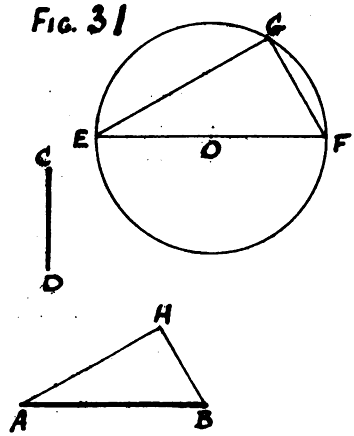


Fig. 32

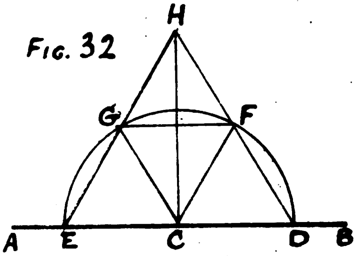


Fig. 33

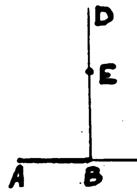


Fig. 34

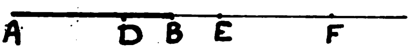


Fig. 35

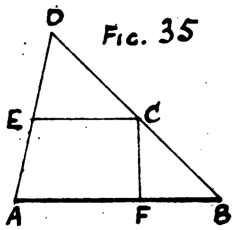


Fig. 36

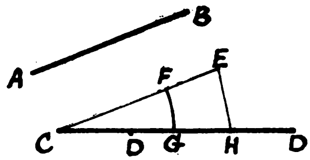


Fig. 37

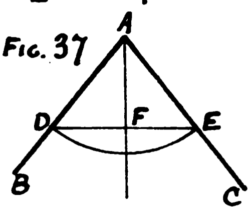


Fig. 38

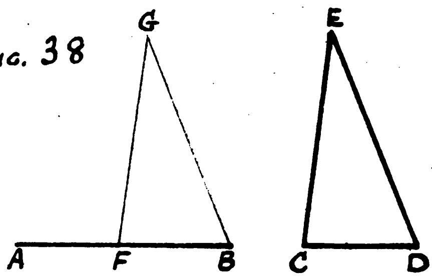


Fig. 39

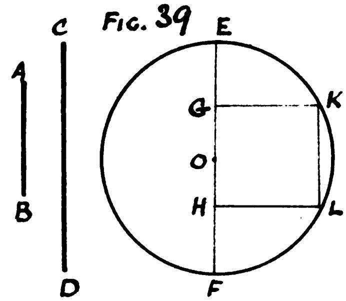


Fig. 40

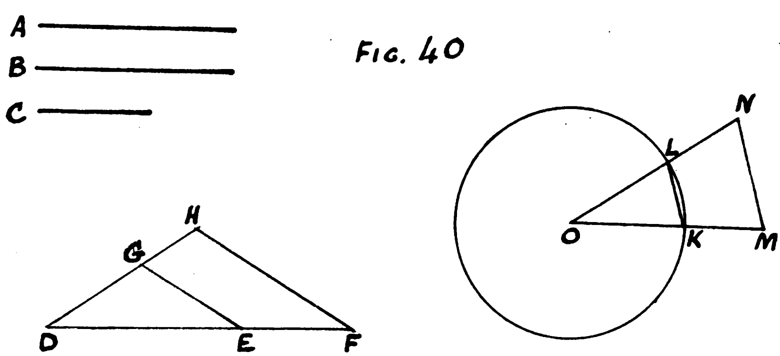


FIG. 41

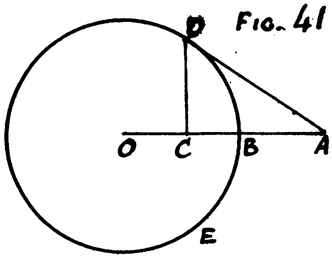


FIG. 42

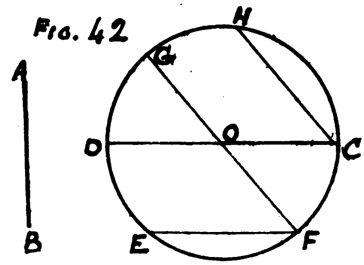


FIG. 43

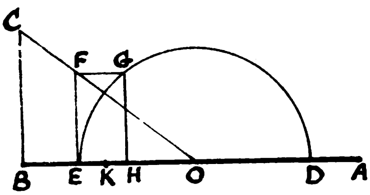


FIG. 44

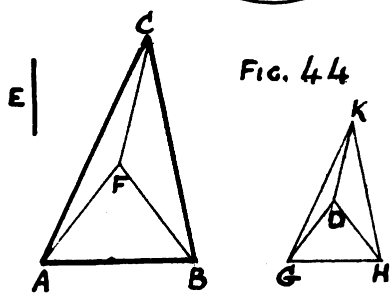


FIG. 45

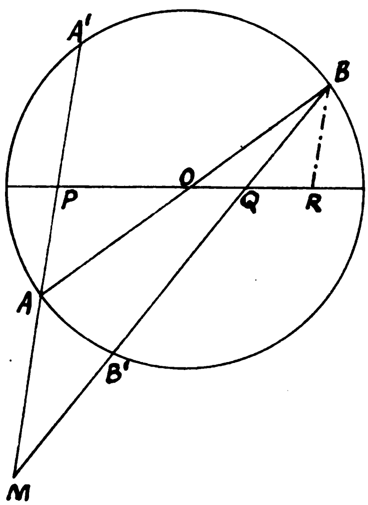


FIG. 46

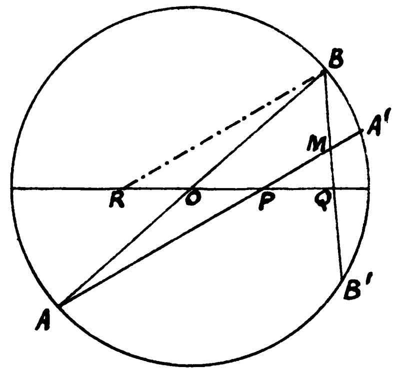


Fig. 47

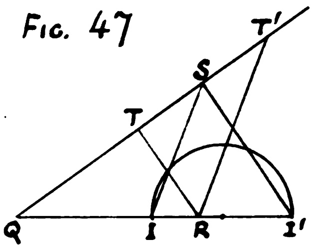


Fig. 48

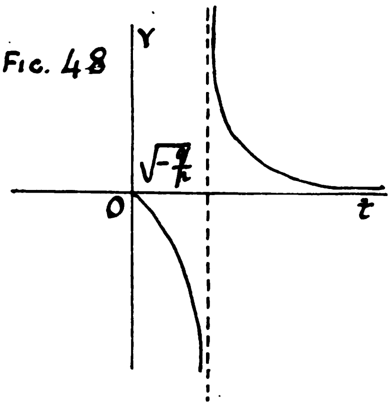


Fig. 49

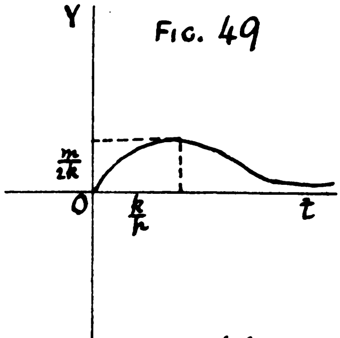


Fig. 50

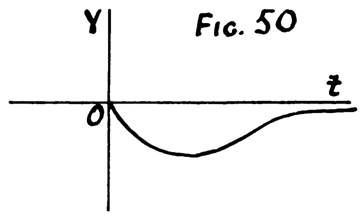


Fig. 51

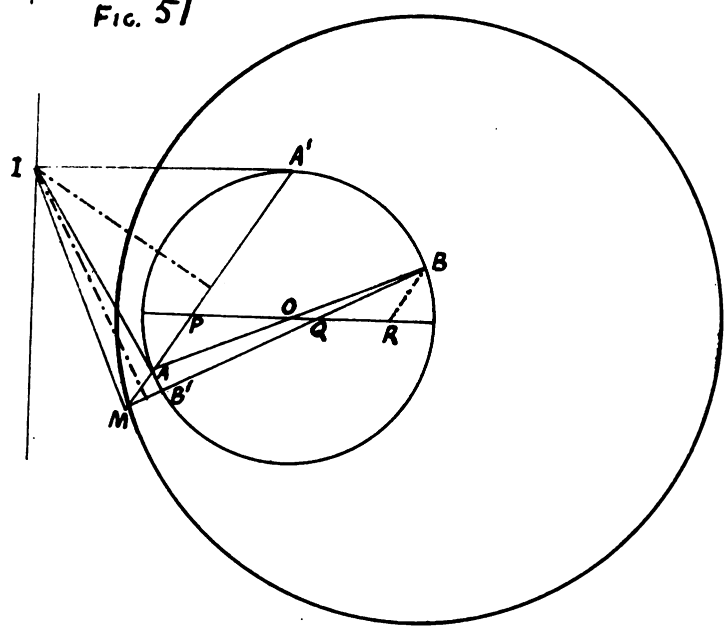


FIG. 52

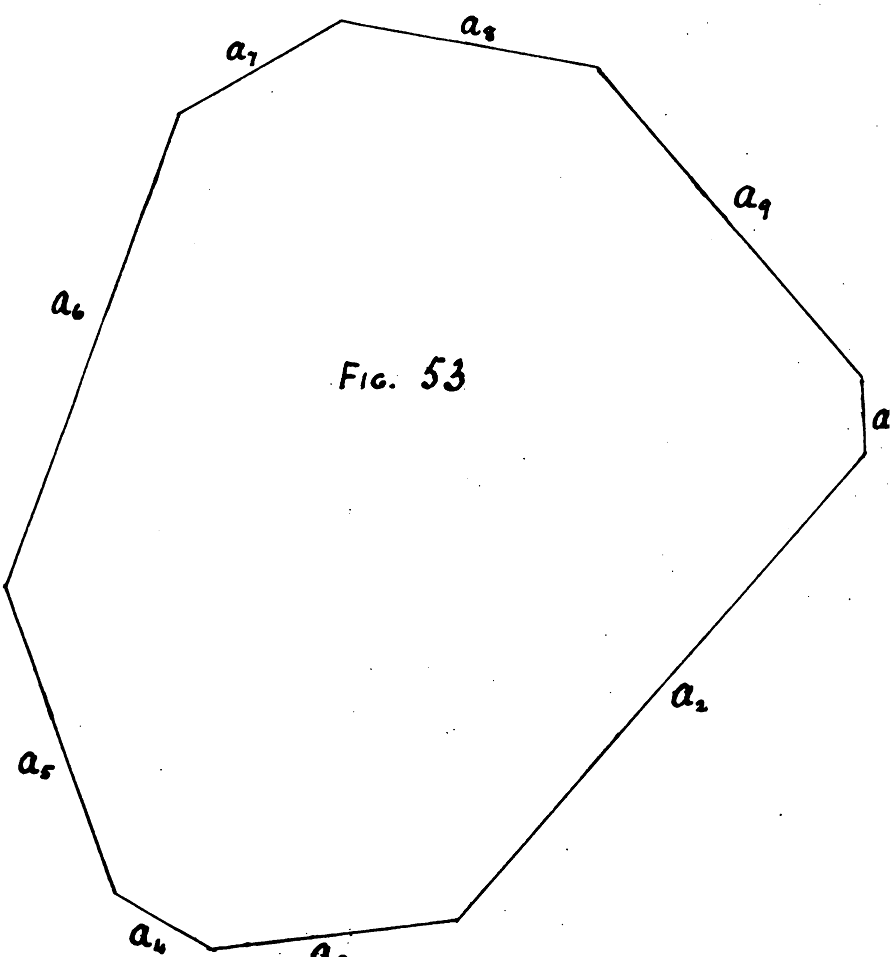


FIG. 53

FIG. 54

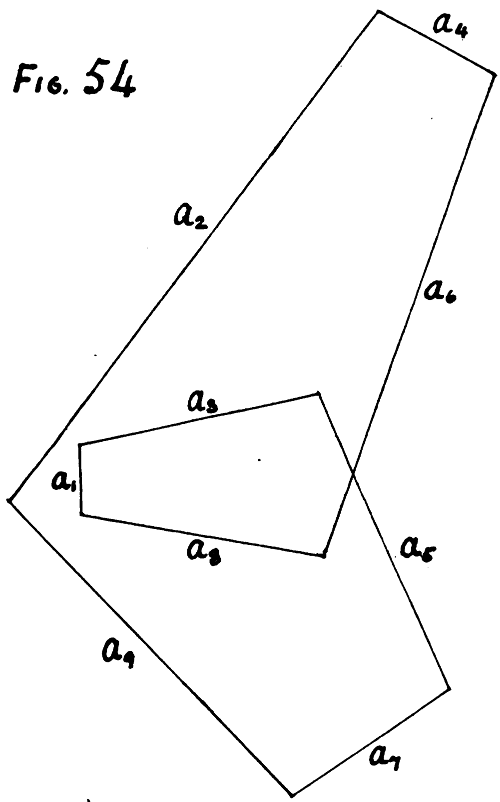


Fig. 55

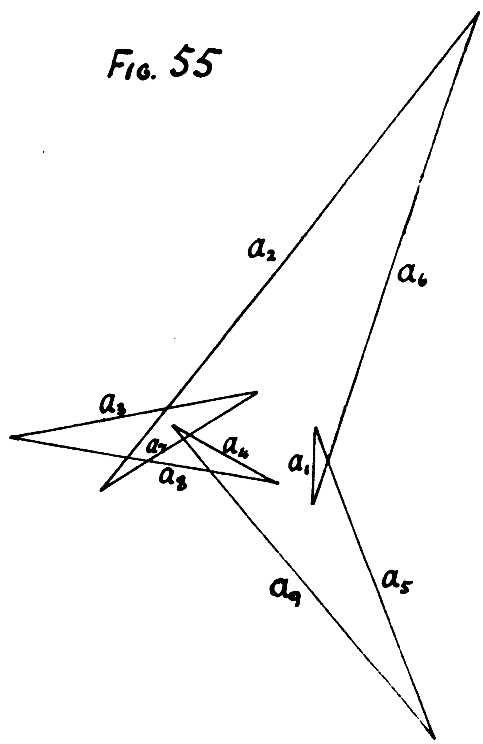


Fig 56

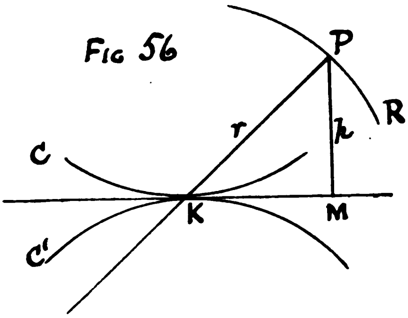


Fig 57

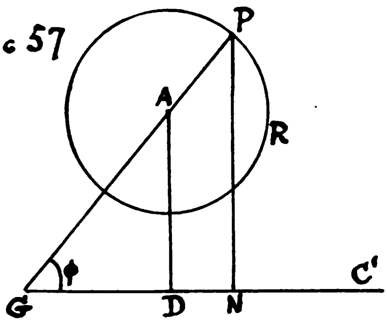


Fig 58

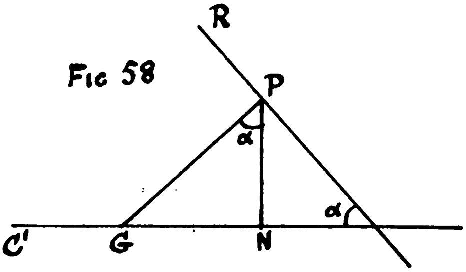


Fig 59

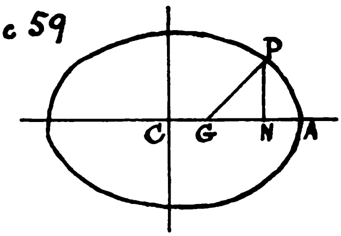


Fig 60

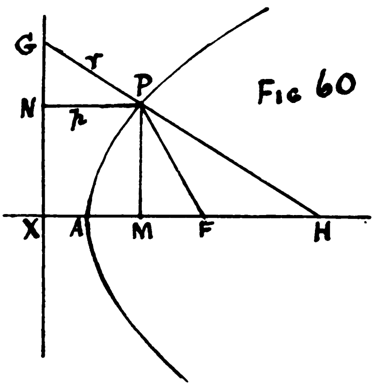


Fig 61

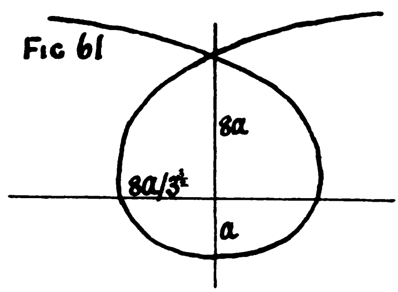


Fig 62

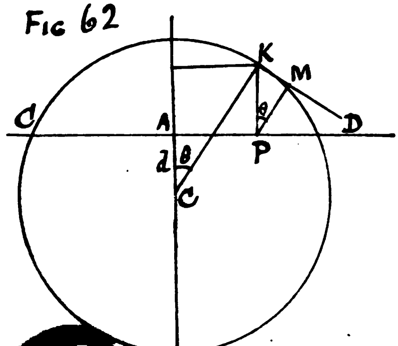


Fig 63

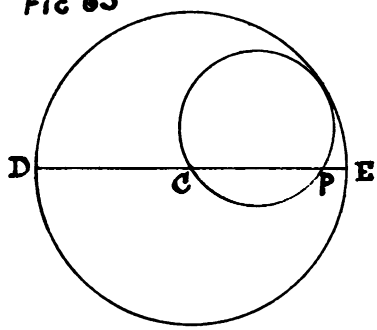


Fig 64

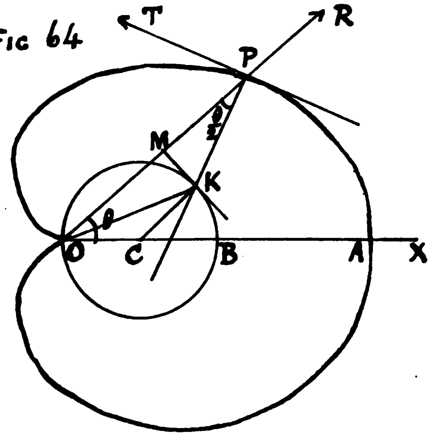


Fig 65

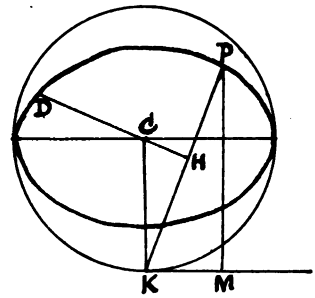


Fig 66

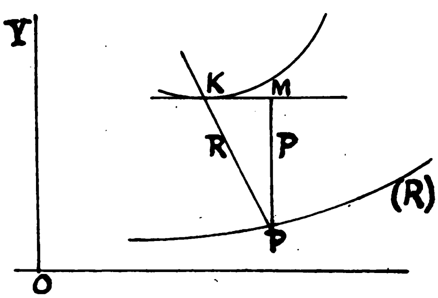


Fig 67

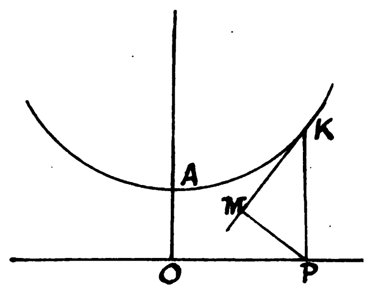


Fig 68

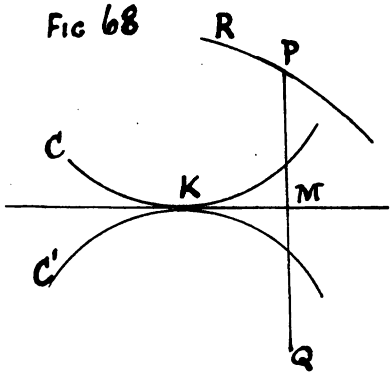


Fig 69

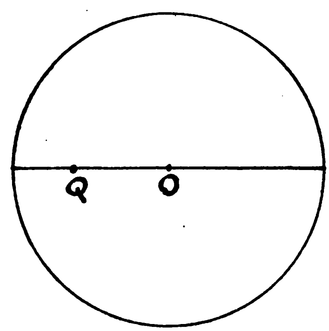


Fig 70

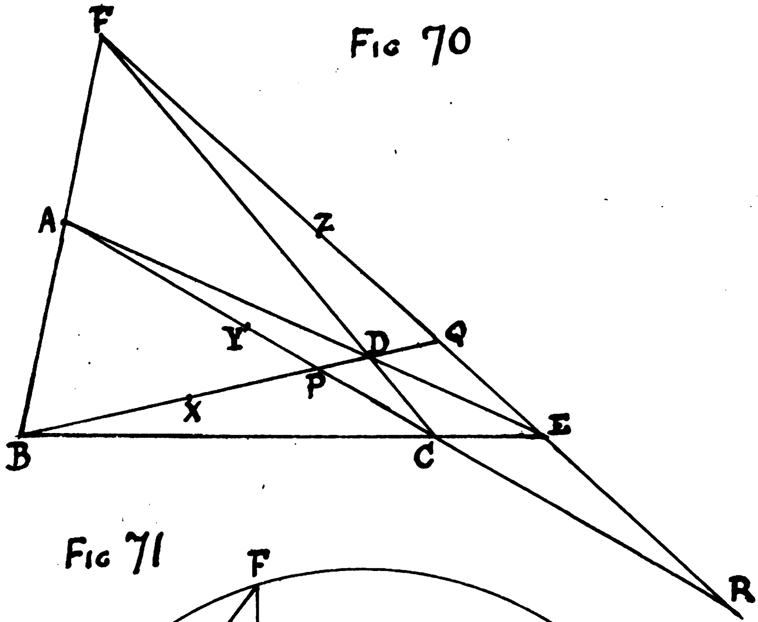


Fig 71

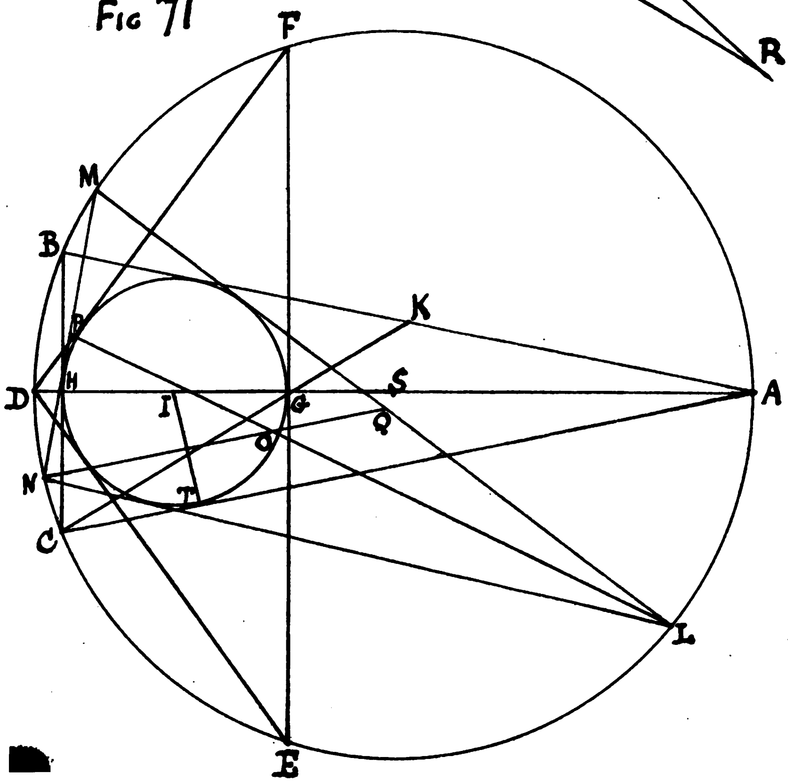


Fig 72

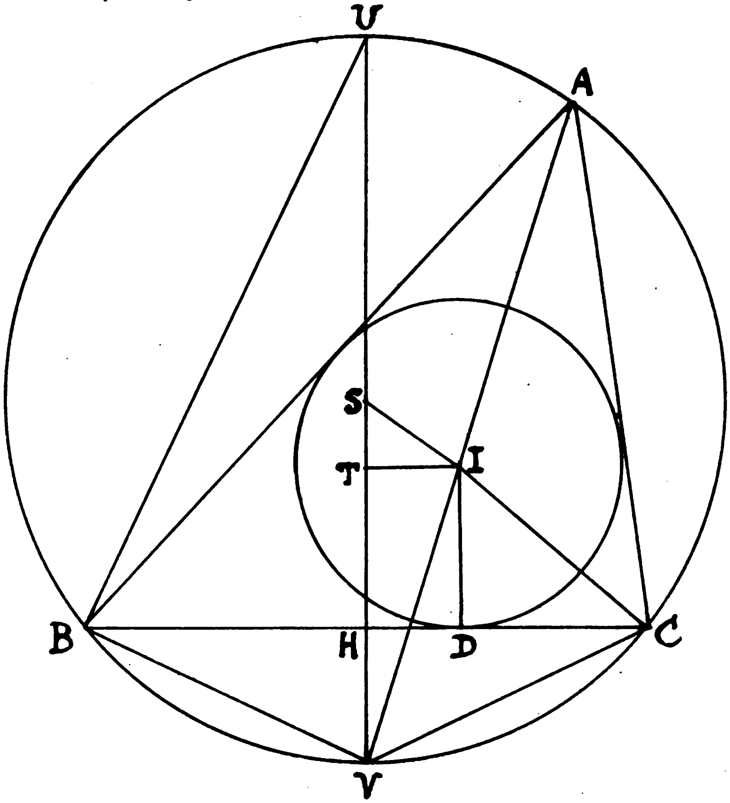


Fig 73

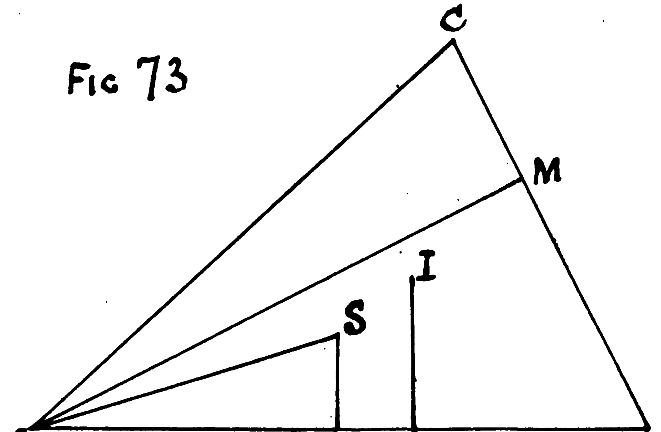


Fig 74

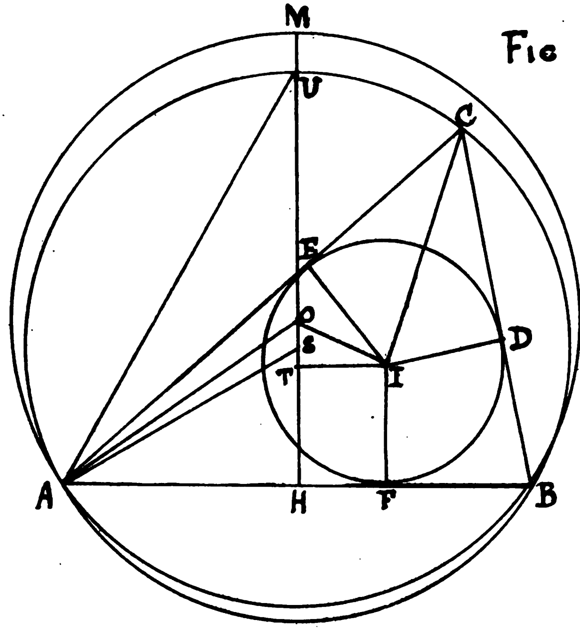


Fig 75

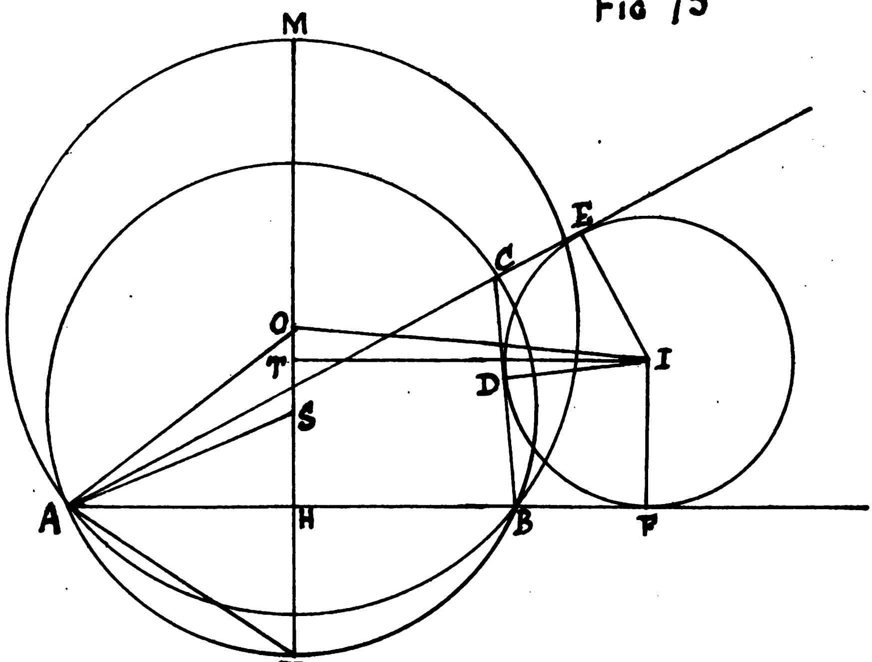


Fig 76

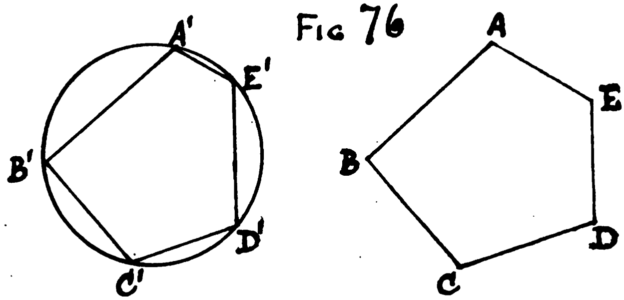


Fig 77

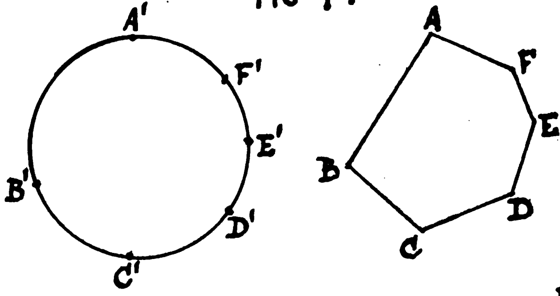


Fig 78

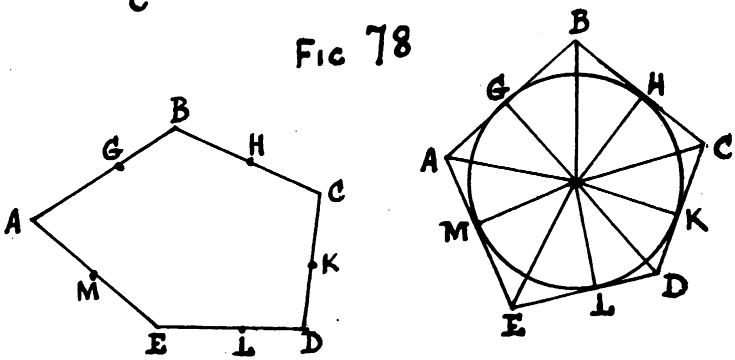


Fig 79

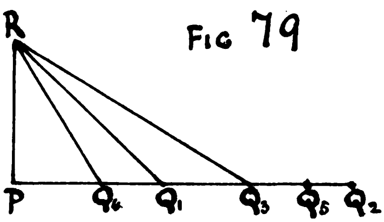


Fig 80

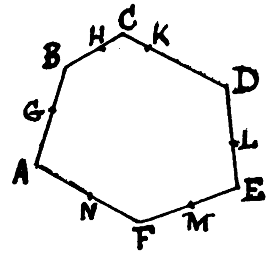


Fig 81

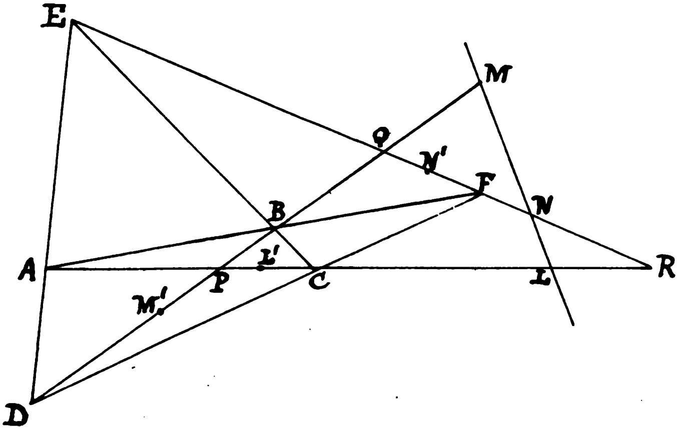


Fig 82

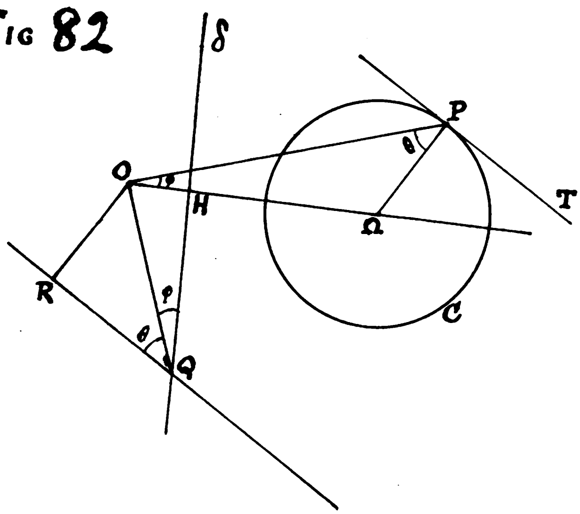


Fig 83

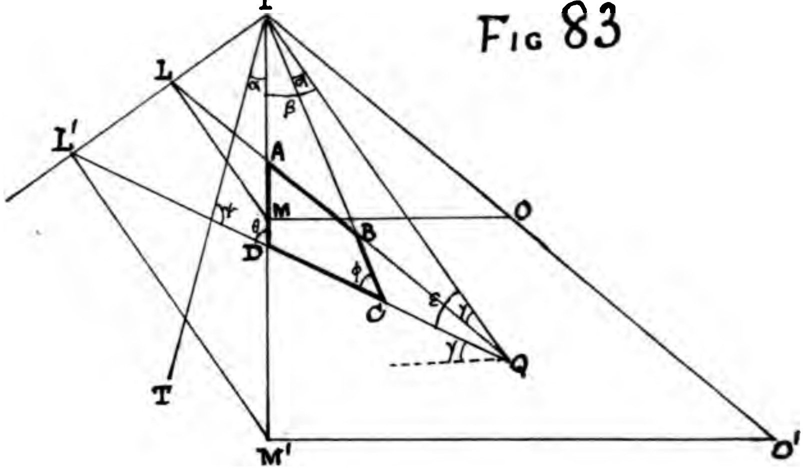
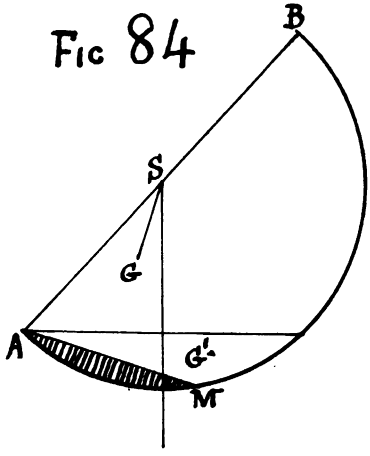
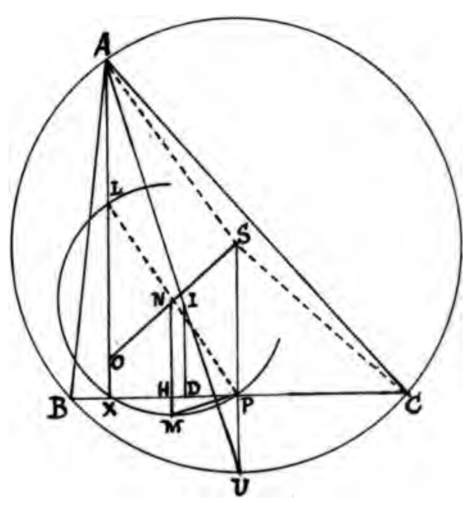


Fig 85

Fig 84





PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME VI.

SESSION 1887–88.

EDINBURGH:
1888.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME VI.

SESSION 1887-88.

EDINBURGH:
1888.

PRINTED BY

JOHN LINDSAY, HIGH STREET, EDINBURGH.

INDEX.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

SIXTH SESSION, 1887-88.

First Meeting, November 11th, 1887.

W. J. MACDONALD, ESQ., M.A., F.R.S.E., Vice-President, in the Chair.

Dr GEORGE THOM, as retiring President, delivered an Address on "Mathematics in Secondary Schools and Universities."

For this Session the following Office-Bearers were elected :—

President,	Mr W. J. MACDONALD, M.A., F.R.S.E.
Vice-President,	Mr GEORGE A. GIBSON, M.A.
Secretary,	Mr A. Y. FRASER, M.A., F.R.S.E.
Treasurer,	Mr JOHN ALISON, M.A.

Committee.

Mr R. E. ALLARDICE, M.A.
Mr ARCHD. C. ELLIOTT, B.Sc.
Mr F. GRANT OGILVIE, M.A. B.Sc.
Mr W. PEDDIE, B.Sc., F.R.S.E.
Dr GEORGE THOM.
Rev. JOHN WILSON, M.A., F.R.S.E.

Of the above, MESSRS ALLARDICE *and* PEDDIE *were elected to edit the Proceedings.*

Properties of the figure consisting of a triangle, and the squares described on its sides.

By J. S. MACKAY, LL.D.

[Of the following properties, some must have been long known, but I do not remember any early statement of them. The 1st, 10th, 15th, 16th, 19th, 20th, and 22nd are given, mostly without proof, in an article by Vecten in Gergonne's *Annales de Mathématiques*, vol. VII., p. 321 (1817); the first parts of the 3rd and 4th are proposed for proof by William Godward in *The Gentleman's Diary* for 1837, p. 48, and proved by him and others in the *Diary* for 1838, pp. 40–41; the first parts of the 5th, 6th, 12th, and 14th are given in M'Dowell's *Exercises on Euclid and in Modern Geometry*, §§ 27, 28, 29 (1863); the 9th in Milne's *Weekly Problem Papers*, p. 135 (1885); the first part of the 8th in Vuibert's *Journal de Mathématiques Elémentaires*, 12^e Année, p. 18 (1887); the 39th in the *Journal de Mathématiques Elémentaires*, edited by De Longchamps, 3^e série, tome I., p. 234 (1887). The others are believed to be new.

For the sake of brevity, I have given only one of the varieties of the figure—that, namely, where the squares are all described outwardly on the sides of the triangle, and I have not mentioned any of the numerous properties that may be derived from the several sets of concurrent straight lines.]

Figure 1.

Let ABC be a triangle; BDEC, CFGA, AHIB the squares described outwardly on its sides. GH, ID, EF are joined, and on GH is described the square GHKL; IK and FL are joined. Then

§ 1. The triangles GAH, IBD, ECF are each equivalent to the triangle ABC.

Since angles BAH, CAG are right, therefore angle GAH is supplementary to angle CAB; and GA, AH are respectively equal to CA, AB; therefore triangle GAH is equal to triangle CAB.

§ 2. The triangles FGL, IHK are each equivalent to the triangle ABC.

For these triangles are each equivalent to the triangle GAH, and therefore to the triangle ABC.

Figure 2.

§ 3. The perpendicular from A to GH is the median from A to BC, and the perpendicular from A to BC is the median from A to GH.

Let AY which is perpendicular to GH meet BC at M, and from B and C draw BP, CQ perpendicular to AM.

Then the right-angled triangles AYH, BPA, having angles HAY, BAP complementary, and $AH = BA$, are congruent; therefore $AY = BP$.

Similarly from the right-angled triangles AYG, CQA there results $AY = CQ$; therefore $BP = CQ$.

Hence the right-angled triangles BMP, CMQ are congruent, and $BM = CM$.

From the relation in which the triangles ABC, AHG stand to each other, it follows from the preceding reasoning that if AX, which is perpendicular to BC, meet HG at N, then $HN = GN$.

Figure 1.

§ 4. The perpendiculars from A to GH, from B to ID, from C to EF intersect at the centroid of the triangle ABC; and the medians from A to GH, from B to ID, from C to EF intersect at the orthocentre of the triangle ABC.

This follows at once from the preceding section.

Figure 2.

§ 5. GH is double of the median from A to BC, and BC is double of the median from A to GH.

The congruent triangles AYG, CQA give $GY = AQ$;
and the congruent triangles AYH, BPA give $HY = AP$;
therefore $GH = GY + HY = AQ + AP$.
But the congruent triangles BMP, CMQ give $MP = MQ$;
therefore $AQ + AP = 2AM$; therefore $GH = 2AM$.

Hence also, as in § 3, $BC = 2AN$.

Figure 1.

§ 6. The triangle whose sides are GH, ID, EF is three times the triangle ABC;
and the triangle whose sides are BC, FL, KI is three times the triangle ABC.

FIG 81

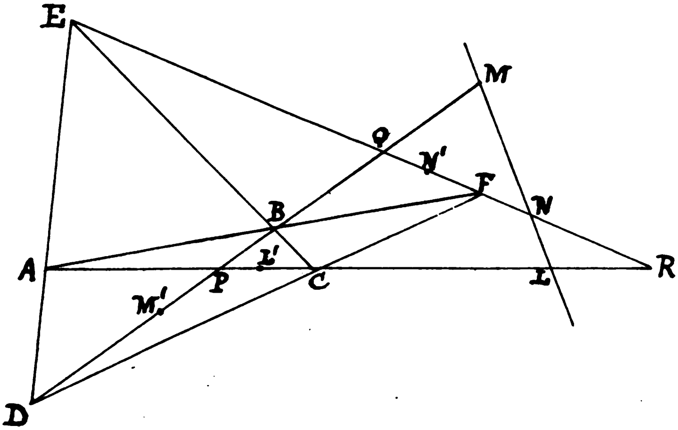
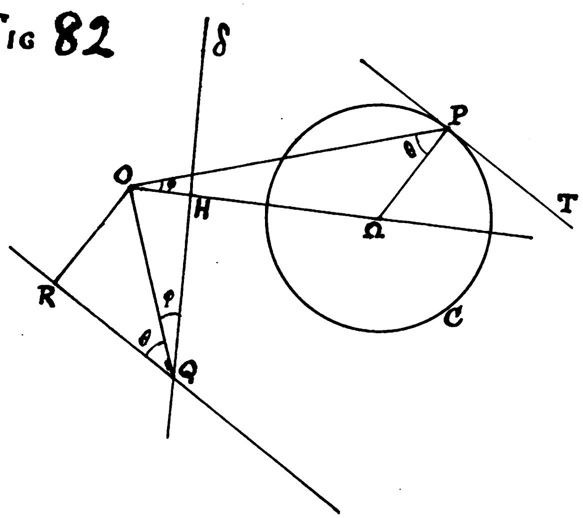


FIG 82



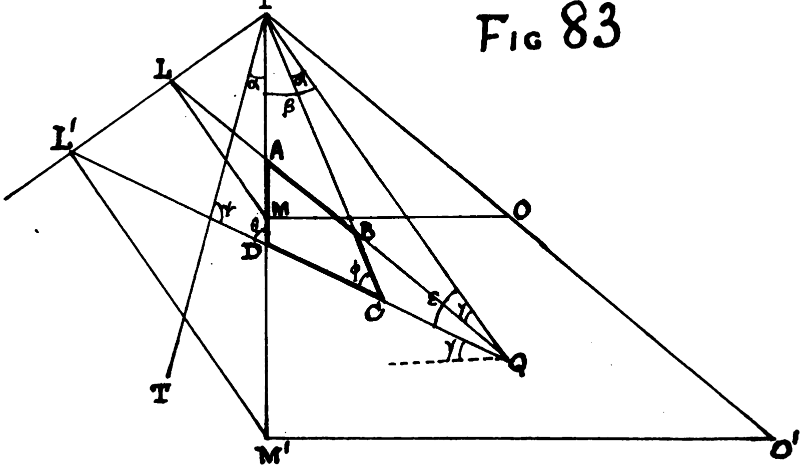


Fig 83

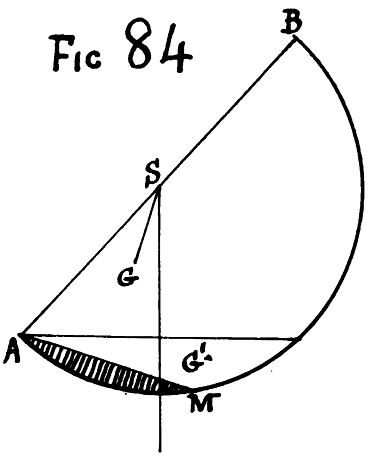
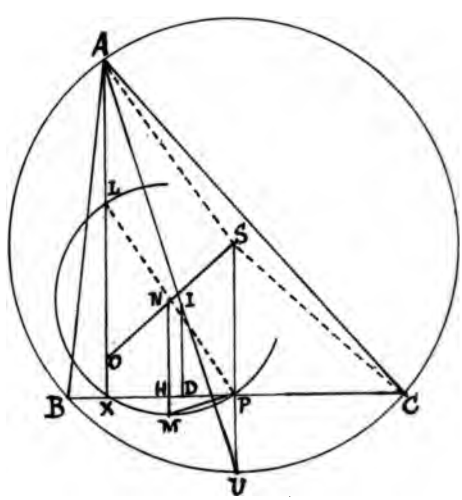


Fig 84

Fig 85



PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME VI.

SESSION 1887–88.

EDINBURGH:
1888.

Because AT_1 is perpendicular to BC, therefore it coincides with the median from A to GH, that is, it passes through N. But BC was proved in § 5 to be double of AN; therefore AT_1 is double of AN.

§ 19. The points P_1, P_2 are on the circle circumscribed about AHIB; P_2, P_3 on the circle BDEC; and P_3, P_1 on the circle CFGA.

Because the angle BP_1H is right, therefore P_1 lies on the circle whose diameter is BH, that is on the circle AHIB; and because the angle AP_2I is right, therefore P_2 lies on the circle whose diameter is AI, that is on the circle AHIB.

§ 20. FI, HE, DG pass respectively through the points P_1, P_2, P_3, and bisect the angles formed by BG and CH, by CI and AD, by AE and BF.

Join FP_1, IP_1.

Because P_1 is a point on the circumference of the circle CFGA, and CF is the chord of a quadrant of that circle, therefore the angle CP_1F is half a right angle. For a similar reason the angle HP_1I is half a right angle; therefore, since CP_1H is a straight line, FP_1I is also a straight line.

Again, the angles CP_1F, GP_1F are each half a right angle; therefore FI bisects the angles formed by BG and CH.

§ 21. AP_1 is perpendicular to FI, BP_2 to HE, and CP_3 to DG.

For the angles AP_1H, HP_1I at the circumference of the circle AHIB stand on arcs each equal to a quadrant.

§ 22. AP_1, BP_2, CP_3 are concurrent at R.

For AP_1 is the common chord or radical axis of the circles AHIB, CFGA; BP_2 that of the circles AHIB, BDEC; CP_3 that of the circles BDEC, CFGA. Hence AP_1, BP_2, CP_3 are concurrent, and R is the radical centre of the three circles.

§ 23. If U, V, W are the centres of the squares described on BC, CA, AB, then VW is parallel to FI and equal to half of it; and similar relations hold between WU and HE, and UV and DG.

For in the triangle AFI, VW joins the middle points of the sides AF, AI; therefore VW is parallel to FI and equal to half of it.

§ 24. AP_1, BP_2, CP_3 pass respectively through U, V, W, and R is the orthocentre of the triangle UVW.

For the angle BP_1C is right, and AP_1 bisects it; therefore AP_1 passes through the centre of the square described on BC.*

Hence UA, VB, WC are the three perpendiculars of the triangle UVW, and R is its orthocentre.

§ 25. If U′ be the centre of the square described on GH, then U, A, U′ are collinear.

For the point P_1 stands in the same relation to the triangle AHG as it does to the triangle ABC, and consequently AP_1 passes also through U′.

§ 26. WU passes through the intersection of CI and BT_1; and UV through the intersection of BF and CT_1.

Since the angles BP_2D, BP_2I are each half a right angle, and since BT_1 is perpendicular to CI, therefore BP_2 is one of the diagonals of a square, three of whose sides coincide with BT_1, CI, AD. Hence the other diagonal of this square, which passes through the intersection of CI and BT_1, will bisect BP_2 perpendicularly. But BP_2 is the common chord of the two circles AHIB, BDEC; therefore it is bisected perpendicularly by WU, the straight line joining the centres. Hence WU passes through the intersection of CI and BT_1.

§ 27. If VW be the side of a square, BG or CH is its diagonal; and a similar relation holds between WU and CI or AD, and between UV and AE or BF.

For $\quad BA : WA = \sqrt{2} : 1$, and $AG : AV = \sqrt{2} : 1$;
therefore in the triangles BAG, WAV, $BA : AG = WA : AV$, and the angle BAG = the angle WAV, because each is equal to the angle BAC increased by a right angle. Hence these triangles are similar, and $BG : WV = \sqrt{2} : 1$.

§ 28. If BG or CH be the side of a square, FI is its diagonal; and a similar relation holds between CI or AD and HE, and between AE or BF and DG.

For $\quad FI : VW = 2 : 1$, and $BG : VW = \sqrt{2} : 1$;

therefore $\quad FI : BG = \sqrt{2} : 1$.

§ 29. The result obtained in the preceding paragraph and several other results may be got in the following way.

Since the points P_1, P_2, P_3 are situated in pairs on the circum-

* See (9) of Mr William Harvey's *Notes on Euclid*, I. 47, in the Proceedings of the Edinburgh Mathematical Society, vol. IV., p. 19.

ferences of the circles circumscribed about the squares, twelve encyclic quadrilaterals are formed, AP_1IH, AP_2IH, BP_1HI, BP_2HI, and two other sets of four corresponding to the sides BC, CA. From two of them BP_1HI and CP_1GF, by the application of Ptolemy's theorem, there result

$$BP_1 \cdot HI + HP_1 \cdot BI = IP_1 \cdot BH,$$

and
$$CP_1 \cdot GF + GP_1 \cdot CF = FP_1 \cdot CG.$$

Now $\quad HI = BI = BH/\sqrt{2}$, and $GF = CF = CG/\sqrt{2}$;

therefore $\quad BP_1 + HP_1 = IP_1 \sqrt{2}, \qquad (\alpha)$

and $\quad CP_1 + GP_1 = FP_1 \sqrt{2}; \qquad (\beta)$

therefore $\quad BG + CH = FI \sqrt{2}$, by addition;

therefore $\quad 2BG = 2CH = FI \sqrt{2}$

therefore $\quad FI : BG = \sqrt{2} : 1.$

§ 30. From the equations (α) and (β), by subtraction, there results

$$(BP_1 - CP_1) + (HP_1 - GP_1) = (IP_1 - FP_1)\sqrt{2}.$$

Now $\quad BP_1 - CP_1 = HP_1 - GP_1$, since $BP_1 + GP_1 = CP_1 + HP_1$;

therefore $\quad 2(BP_1 - CP_1) = 2(HP_1 - GP_1) = (IP_1 - FP_1)\sqrt{2}$,

or $\quad IP_1 - FP_1 = (BP_1 - CP_1)\sqrt{2} = (HP_1 - GP_1)\sqrt{2}$.

Similar relations hold for $EP_2 - HP_2$ and $GP_3 - DP_3$.

§ 31. From the encyclic quadrilaterals AP_1IH and AP_1FG there result

$$AP_1 + IP_1 = HP_1 \sqrt{2},$$

and
$$AP_1 + FP_1 = GP_1 \sqrt{2};$$

therefore $\quad 2AP_1 + FI = (HP_1 + GP_1)\sqrt{2}$, by addition.

Similar relations hold for $2BP_2 + HE$, and $2CP_3 + DG$.

§ 32. $AU = VW$, $BV = WU$, $CW = UV$.

Because $\quad (GP_1 + HP_1)\sqrt{2} = 2AP_1 + FI$,

therefore $\quad GP_1 + HP_1 = AP_1 \sqrt{2} + FI/\sqrt{2}$,

$\quad = AP_1 \sqrt{2} + BG$.

Now $\quad BP_1 + CP_1 = UP_1 \sqrt{2}$*;

therefore $\quad BG + CH = AU \sqrt{2} + BG$, by addition;

therefore $\quad CH = AU \sqrt{2}$.

But $\quad CH = VW \sqrt{2}$; therefore $AU = VW$.

---

* See (5) of Mr Harvey's paper, before referred to.

This result will be established more simply later on.

§ 33. The perpendiculars from M and N to BC and GH intersect each other at the middle point of FI.

Because M is the middle point of F′I′, and FF′, II′ are perpendicular to BC, therefore the perpendicular to BC at M will bisect FI. For a similar reason the perpendicular to GH at N will bisect FI.

§ 34. If Z be the middle point of FI, then ZBUC and ZHU′G are squares.

For $ZM = \frac{1}{2}(FF' + II') = \frac{1}{2}BC$; and UM is perpendicular to BC and equal to $\frac{1}{2}BC$; therefore ZBUC is a square.

For a similar reason ZHU′G is a square.

§ 35. AMZN is a parallelogram whose diagonals intersect at the middle point of VW.

For $ZM = \frac{1}{2}BC = AN$, and $ZN = \frac{1}{2}GH = AM$; therefore AMZN is a parallelogram. Hence MN bisects AZ. But VW joins the middle points of two sides of the triangle AFI; therefore VW bisects, and is itself bisected by, the median AZ.

§ 36. If O_1 is the middle point of AZ, then O_1M is parallel to AU and equal to half of it.

For in the triangle ZAU, O_1 is the middle point of ZA and M is the middle point of ZU.

Similarly O_1N is parallel to AU′ and equal to half of it. Hence $UU' = 2MN$.

§ 37. The middle point of any side of a triangle is equidistant from the centres of the squares described in the same manner on the other two sides.

For O_1 is the middle point of VW and O_1M, which is parallel to AU, is perpendicular to VW; therefore $MV = MW$.

§ 38. If V′, W′ be the projections of V, W on BC, then $VV' = XV'$ and $WW' = XW'$.

For the quadrilateral AXCV is encyclic, since the angles AXC, AVC are each right; therefore the angle VXC = the angle VAC = half a right angle; therefore $VV' = XV'$. Similarly $WW' = XW'$.

§ 39. The centroid of the triangle UVW is the same as the centroid of the triangle ABC, and that of U′VW the same as that of AHG.

The distance of the centroid of the triangle UVW from BC is $\frac{1}{3}(VV' + WW' - UM)$, and the distance of the centroid of the

triangle ABC from BC is $\frac{1}{3}$ AX. It remains, therefore, to prove that $VV' + WW' - UM = AX$.

From the encyclic quadrilateral AXCV, by application of Ptolemy's theorem, there results

$$AX \cdot CV + CX \cdot AV = VX \cdot AC.$$

Now $\qquad CV = \quad AV = AC/\sqrt{2}\,;$

therefore $\qquad AX + CX \quad = VX\sqrt{2} = 2VV'.$

Similarly $\qquad AX + BX \quad = 2WW'.$

But $\qquad BC \quad = 2UM\,;$

therefore $\qquad 2(VV' + WW' - UM) = 2AX + BX + CX - BC$

$$= 2AX.$$

Hence also the distance of the centroid of UVW from either AB or CA is the same as the distance of the centroid of ABC from AB or CA. The two triangles consequently have the same centroid.

§ 40. O_1 is the centre of a circle which passes through the following ten points :—V, W, M, X, N, Y, the feet of the perpendiculars from V on WU, WU′, and from W on VU, VU′.

The circle with O_1 as centre and OV or OW as radius is readily seen to pass through the feet of the four perpendiculars from V and W. Also this circle will pass through N and Y, if it can be shown to pass through M and X.

Now O_1M is half of AU, and $AU = VW$; therefore O_1M is half of VW ; therefore the circle passes through M.

But since AX and ZM are perpendicular to BC, and $O_1A = O_1Z$, therefore $O_1X = O_1M$; and therefore the circle passes through X.

[The circle on VW as diameter can be proved to pass through X thus : the angle VXC is half a right angle, by § 38, and so is the angle WXB ; therefore the angle VXW is a right angle.]

If O_2, O_3 be the middle points of WU, UV then, O_2, O_3 will be the centres of two other ten point circles.

The Potential of a Spherical Magnetic Shell deduced from the Potential of a Coincident Layer of Attracting Matter.

By A. C. Elliott, B.Sc., C.E.

This is the problem of § 670 in Clerk Maxwell's *Electricity and Magnetism.* The author proposes to proceed by another method and to obtain the result in a different form. Let O be the centre of the spherical surface on which the shell lies and Z the point where the

magnetic potential V_m is to be found. Also let ϕ be the strength of the shell (magnetic moment per unit area), a its internal, and $a+\delta a$ its external radius. To represent the magnetic distribution let a layer of negative magnetic matter of density σ cover the inside face, and a corresponding positive layer the outside face. Finally, let Z be without the matter of the shell and on the positive side.

Since in a magnet the total quantity of magnetic matter is zero, these hypothetical layers are subject to the condition

$$a^2\sigma = \text{const.} \qquad (1).$$

Let V be the potential at Z due to a single layer of density σ and radius a. The magnetic potential V_m is the sum of the potentials due to the two imaginary layers; and hence by Taylor's theorem

$$V_m = V + \frac{dV}{da}\delta a + \frac{dV}{d\sigma}\delta\sigma - V$$

$$= \frac{dV}{da}\delta a + \frac{dV}{d\sigma}\delta\sigma \qquad (2).$$

From the nature of the potential function

$$V = A\sigma \qquad (3),$$

where A is independent of σ—in fact, the potential for unit density.

From (1) $$\delta\sigma = -\frac{2\sigma}{a}\delta a$$

From (3) $$\frac{dV}{d\sigma} = A = \frac{V}{\sigma}$$

Therefore (2) becomes

$$V_m = \frac{dV}{da}\delta a - \frac{2V}{a}\delta a$$

or since δa is an independent variation

$$V_m = a^2\frac{d}{da}\left(\frac{V\delta a}{a^2}\right) \qquad (4).$$

But $$V\delta a = A\sigma\delta a = A\phi.$$

Hence if P be the potential at Z due to a layer of density numerically equal to ϕ

$$V_m = a^2\frac{d}{da}\left(\frac{P}{a^2}\right) \qquad (5).$$

Calling r the distance OZ, Maxwell obtains

$$V_m = -\frac{1}{a}\frac{d}{dr}(Pr) \qquad (6).$$

It appears therefore that the operations denoted by (5) and (6) respectively are equivalent. The first might sometimes be the more convenient to use—for instance, Maxwell, § 695, eqn. 6′.

Second Meeting, December 9th, 1887.

W. J. Macdonald, Esq., M.A., F.R.S.E., President, in the Chair.

A Method of Transformation in Geometry.

By R. E. Allardice, M.A.

In spherical, as in plane geometry, transformation by inversion and by similar figures may be used for the deduction of new theorems from known ones, the two methods, however, in the former case becoming identical; and like all other propositions and methods in spherical geometry, these methods of transformation may be dualized. The transformation indicated as the dual of these in the case of the spherical surface is also applicable *in plano*. The sequel is an account of this method and of some results that may be obtained by means of it.

Let XY (fig. 5) be a fixed straight line, P the curve to be transformed, AB a tangent to P meeting XY in B and making with XY an angle θ, BC a straight line making with XY an angle ϕ; then if θ and ϕ are connected by the relation $\tan\frac{1}{2}\phi = k\tan\frac{1}{2}\theta$, the envelope of BC is the transformed curve. This transformed curve may be called, for reference, the second inverse of P. [The phrase "second inverse" of P would naturally mean the inverse of the inverse of P; but, as this is the curve P itself, the phrase is not much used in this sense, and may therefore be employed to designate the curve obtained from P by the method of tranformation that is dual to the method of inversion.]

The second inverse of a circle is another circle such that the axis of transformation is the radical axis of the two circles.

This theorem is most easily established by proving first the converse, which is as follows :—

Let P (fig. 6) be a point in the radical axis of two circles; PQ and PR tangents to these circles; to prove that the ratio $\tan\frac{1}{2}\text{QPE}$: $\tan\frac{1}{2}\text{RPE}$ is constant, independent, that is, of the position of P.

Let PC and PD, the bisectors of the angles at P, meet the central axis of the circles in C and D respectively; then the theorem is equivalent to this, that the ratio CE : ED is constant.

Let QA and PC meet in O.

Since $\angle \text{CPQ} = \angle \text{CPE}$, $\therefore$ $\text{AC} = \text{AO}$.

Now $$\text{CE} : \text{EP} = \text{QO} : \text{QP},$$

$\therefore$ $$\text{CE} : \text{CE} + \text{QA} + \text{AC} = \text{EP} : \text{EP} + \text{QP},$$

$\therefore$ $$\text{CE} : \text{EH} = \text{EP} : \text{EP} + \text{QP};$$

similarly $$\begin{aligned}\text{DE} : \text{EK} &= \text{EP} : \text{EP} + \text{RP}\\ &= \text{EP} : \text{EP} + \text{QP}.\end{aligned}$$

$\therefore$ $$\begin{aligned}\text{CE} : \text{DE} &= \text{EH} : \text{EK}\\ &= \text{constant}.\end{aligned}$$

The converse of this theorem is obviously true; that is, if the circle A and the line PE be given and if PR be always drawn so that $\tan\frac{1}{2}\text{RPE} = k\tan\frac{1}{2}\text{QPE}$, where k is a constant, then PR will envelope a circle, PE will be the radical axis of this circle and the given circle, and the ratio EK : EH will be equal to k.

It may be noticed here that although the transformation was originally defined analytically, a purely geometrical form may be given to it; for (fig. 6) the conditions

$$\tan\frac{1}{2}\text{RPE} = k\tan\frac{1}{2}\text{QPE} \text{ and } \text{CE} = k\text{ED} \text{ are identical.}$$

Expression for the radius of the transformed circle.

In fig. 6 let $\text{AE} = d$, $\text{AH} = r$, $\text{BE} = d'$, $\text{BK} = r'$; then by the property of the radical axis

$$\text{EK}.\text{EK}' = \text{EH}.\text{EH}',$$

but $$\text{EK} = k\text{EH},$$

$\therefore$ $$\text{EK}' = \text{EH}'/k.$$

Hence $$\begin{aligned}d' + r' &= k(d+r),\\ d' - r' &= (d-r)/k;\end{aligned}$$

$\therefore$ $$2r' = r(k + 1/k) + d(k - 1/k).$$

Hence the radius of the transformed circle will be zero, or this circle will reduce to a point, if

$$k^2 = (d - r)/(d + r),$$

which gives real values for k if the given circle does not cut the axis of transformation.

Another demonstration may be obtained by using a special kind of co-ordinates. Thus (fig. 7) consider a circle of radius a, the distance of the centre of which from a fixed axis is d. Let the tangent at any point P make an angle θ with the axis, and let r be the length of the part of this tangent intercepted by the axis; then

$$r\sin\theta = d + a\cos\theta,$$
$$\therefore \quad r = d\operatorname{cosec}\theta + a\cot\theta,$$

is the equation to the circle in this system of co-ordinates.

This equation may be written

$$r = d\left(1 + \tan^2\frac{1}{2}\theta\right)/2\tan\frac{1}{2}\theta + a\left(1 - \tan^2\frac{1}{2}\theta\right)/2\tan\frac{1}{2}\theta$$

or, $$r = \left\{(d + a) + (d - a)\tan^2\frac{1}{2}\theta\right\}/2\tan\frac{1}{2}\theta.$$

The equation is transformed by writing

$$k\tan\frac{1}{2}\phi \text{ in place of } \tan\frac{1}{2}\theta.$$

This gives $$r = \left\{(d + a) + (d - a)k^2\tan^2\frac{1}{2}\phi\right\}/2k\tan\frac{1}{2}\phi$$

or $$r = \left\{(d + a)/k + (d - a)k\tan^2\frac{1}{2}\phi\right\}/2\tan\frac{1}{2}\phi,$$

which represents a circle. If δ be the distance of its centre from the axis and α its radius, then

$$\delta + \alpha = (d + a)/k,$$
$$\delta - \alpha = (d - a)k,$$

results which agree with those obtained by the former method.

The principal characteristic of this method of transformation is that the length of the tangent from a point in the axis remains unaltered. This is obvious in the case of the circle from the demonstration given above; and, in the case of any other curve, may be seen from the fact that, if a circle be drawn to touch the curve at its point of contact with a given tangent, and the whole figure be transformed, the transformed circle will touch the other transformed

curve at its point of contact with the line into which the given tangent transforms. A direct proof of this property by means of infinitesimals may also be given, as follows:—

Let AP, BP, (fig. 8) be two consecutive tangents; AQ, BQ, the transformed tangents; let $PAX=\theta$, $QAX=\phi$; draw AN and AM perpendicular to BP and BQ respectively; then

$$AP = AN/APB = AN/d\theta = AB\sin\theta/d\theta.$$

Now, $\tan\frac{1}{2}\phi = k\tan\frac{1}{2}\theta$, $\therefore\ \sec^2\frac{1}{2}\phi d\phi = k\sec^2\frac{1}{2}\theta d\theta$;

$$\begin{aligned}\therefore\ AP &= AB\sin\theta/d\theta\\ &= 2AB\tan\frac{1}{2}\theta/\sec^2\frac{1}{2}\theta d\theta\\ &= 2AB\tan\frac{1}{2}\phi/\sec\frac{1}{2}\phi d\phi\\ &= AQ.\end{aligned}$$

If a tangent to a given curve P makes an angle θ with a given axis, it may also be considered as making an angle $\pi+\theta$ with that axis. Hence if the angle of inclination ϕ of the transformed tangent is determined by the relation $\tan\frac{1}{2}\phi = k\tan\frac{1}{2}\theta$, there are two possible values of ϕ, and therefore two possible positions of the transformed tangent. Each of these tangents has an envelope, and hence a curve may, in general, be transformed in two different ways for the same value of k. In the case of the circle, these two transformations may easily be discussed geometrically; the following is an analytical investigation.

Let the co-ordinates of any point on the circle (figs. 9 and 10) be expressed in terms of the angle θ contained by a radius to the point and a radius parallel to the axis of transformation, namely

$$x = a + r\cos\theta,\ y = b + r\sin\theta.$$

Let ϕ be the angle made by a tangent to the circle with the axis of transformation, and let $\tan\frac{1}{2}\psi = k\tan\frac{1}{2}\phi$.

In fig. 9, $\phi = \theta + \frac{\pi}{2}$; in fig. 10, $\phi = \theta + \frac{3\pi}{2}$. Consider fig. 9.

$$\begin{aligned}x &= a + r\cos\theta = a + r\sin\phi\\ y &= b + r\sin\theta = b - r\cos\phi\end{aligned}$$

The equation to the tangent at (x_1, y_1) is

$$y - y_1 = (x - x_1)\tan\phi.$$

When $y=0$ $\qquad x'=x_1-y_1\cot\phi$;
and the transformed curve is the envelope of

$$y=(x-x')\tan\psi,\text{ where }\tan\frac{1}{2}\psi=k\tan\frac{1}{2}\phi;$$

that is

$$\begin{aligned}y&=(x-x_1+y_1\cot\phi)\tan\psi\\&=(x-a+b\cot\phi-r\operatorname{cosec}\phi)\tan\psi.\end{aligned}$$

If $\tan\frac{1}{2}\phi=\lambda$, this equation reduces to

$$\{k^2y-(r+b)k\}\lambda^2+2(x-a)k\lambda+(b-r)k-y=0;$$

the envelope of which is given by

$$(x-a)^2k^2-\{k^2y-(r+b)k\}\{(b-r)k-y\}=0$$

that is,

$$(x-a)^2+\left\{y-\frac{1}{2}b(k+1/k)+\frac{1}{2}r(k-1/k)\right\}^2=\frac{1}{4}\{b(k-1/k)-r(k+1/k)\}^2.$$

To get the equation for the case of fig. 10, the sign of r must be changed throughout.

Let δ and ρ be the distance of the centre from the axis and the radius for fig. 9, and δ' and ρ' the corresponding quantities for fig. 10. Then

$$\delta=\frac{1}{2}b(k+1/k)-\frac{1}{2}r(k-1/k);\ \delta'=\frac{1}{2}b(k+1/k)+\frac{1}{2}r(k-1/k);$$

$$\rho=\frac{1}{2}b(k-1/k)-\frac{1}{2}r(k+1/k);\ \rho'=\frac{1}{2}b(k-1/k)+\frac{1}{2}r(k+1/k).$$

$$\delta+\rho=(b-r)\,k;\qquad \delta'+\rho'=(b+r)k;$$

$$\delta-\rho=(b+r)/k;\qquad \delta'-\rho'=(b-r)/k.$$

Hence, for any given value of k, a circle A may be transformed into two distinct circles B and C; the difference being that for one of the circles, B say, the tangent which envelopes A is supposed to move round A in the positive direction, while for C the enveloping tangent moves round A in the negative direction. Only one of the circles B and C may be considered at a time, or both at once, as is convenient. The two forms of transformation may be distinguished as direct and inverse.

In fig. 11, A and B are two circles whose direct common tangents intersect in the axis of transformation. A transforms into the two circles C and C′, and B into D and D′. As is shown in the figure, a direct common tangent to A and B transforms into a direct common tangent to C and D, and also into a direct common tangent to C′ and D′; while an inverse common tangent to A and B transforms

into a common tangent to C and D′ and also into a common tangent to C′ and D; that is to say, if two circles are transformed either both directly or both inversely, the *direct* common tangents transform into common tangents, while if one of the circles is transformed directly and the other inversely, the *inverse* common tangents transform into common tangents. It is obvious that when two circles are transformed the length of the common tangent remains unaltered. This corresponds to the fact that, in transformation by similar figures, the angle of intersection of two curves remains unaltered.

It was shown that, by selection of a proper value of k, any circle may be transformed into a point. In this case any two tangents from a point in the axis of transformation transform into the same straight line; and hence any two circles, the common tangents to which intersect in the axis of transformation may be transformed into points by the same transformation. If an axis of similitude is taken for axis of transformation, any three circles may be transformed into points; and in general any number of circles which possess a common axis of similitude may be so transformed, two circles that lie on the same side of the axis being transformed either both directly or both inversely, and two that lie on opposite sides being transformed, one directly and one inversely. It may be noted that a point transforms into only one circle.

If a single circle is transformed in all possible ways, that is, for all values of k, an infinite series of circles is obtained, consisting of all the circles that cut another system orthogonally. If this system is transformed for any one value of k, it transforms into itself.

If all the points on the circumference of a circle be transformed, the centres of the circles into which they transform will lie on an ellipse.

Let P (fig. 12) transform into a circle of which the centre is P′, and which passes through Q and R,

then

$$\mathrm{QM} = \mathrm{PM}/k \text{ and } \mathrm{RM} = k\mathrm{PM}$$
$$\therefore\ \mathrm{QM} + \mathrm{RM} = (k + 1/k)\mathrm{PM}$$
$$\therefore\ \mathrm{P'M} = \frac{1}{2}(k + 1/k)\mathrm{PM}.$$

Hence the locus of P′ is an ellipse, one of the axes of which is equal to the diameter of the circle on which P lies. The other axis is $\frac{1}{2}(k + 1/k)$ times this diameter.

Examples of the Application of this Method.

(1) By means of this method of transformation may be found very easily the condition that four straight lines touch the same circle, in terms of the angles that the lines make with the third diagonal of the quadrilateral formed by them.

Let AB (fig. 13) be this third diagonal; and let the circle touched by the four lines be transformed into the point P.

Let PA and PB make angles α and β with AB.

Let the two tangents through A make angles ϕ_1 and ϕ_2, and the two tangents through B angles ψ_1 and ψ_2, with AB.

Then $$\tan\frac{1}{2}\alpha = k\tan\frac{1}{2}\phi_1 \qquad \tan\frac{1}{2}\beta = k\tan\frac{1}{2}\psi_1$$

$$\tan\frac{1}{2}(\alpha+\pi) = k\tan\frac{1}{2}\phi_2 \qquad \tan\frac{1}{2}(\beta+\pi) = k\tan\frac{1}{2}\psi_2.$$

The required condition is obtained at once, by elimination of α, β and k from these four equations; namely,

$$*\tan\frac{1}{2}\phi_1 \tan\frac{1}{2}\phi_2 = \tan\frac{1}{2}\psi_1 \tan\frac{1}{2}\psi_2.$$

(2) If four circles have a common axis of similitude, they may be transformed into four points. Hence their six common tangents satisfy the relation which connects the six straight joining four points.

It should be noted here, and the remark applies to all cases in which an axis of similitude and a common tangent are referred to, that when the axis of similitude passes through the external centre of similitude of two circles, the direct common tangent is to be understood, and when through the internal centre of similitude the inverse common tangent.

(3) If four circles have a common axis of similitude, and touch a common circle, their common tangents satisfy the relation connecting the diagonals and sides of a quadrilateral inscribed in a circle, namely,

$$h^2 = (ac+bd)\ (ad+bc)/(ab+cd),\ k^2 = (ac+bd)\ (ab+cd)/(ad+bc).$$

(4) Given three fixed circles A, B, C, the locus of the centre of a circle P such that the common tangents to P and A, P and B, P

---

* This result is implicitly contained in a paper by me in vol. iii. of the *Proceedings*, page 60.

and C, are equal is a straight line perpendicular to the axis of similitude of the circles A, B, C.

This result is obtained by transforming the circle A, B, C, into three points, and noticing that the system of circles P transforms into a system of concentric circles.

There are four different cases, one arising from consideration of each of the axes of similitude.

The common tangent to either of the circles A, B, C, and the circle P is a maximum when P, A, B, C have a common axis of similitude.

(5) The eight circles touching the circles A, B, C, of the last example are particular cases of the circle P. Hence the centres of these eight circles lie in pairs on straight lines, passing through the radical centre and perpendicular to the axes of similitude of the circles A, B, C.

The two circles that have their centres in the same line perpendicular to one of the axes of similitude have obviously that axis for their radical axis. (See *Casey's Sequel to Euclid*, Third Edition, page 121.)

(6) The last example gives a method for constructing the circles touching three given circles.

(7) Let three points be transformed with every possible value of k. A one-fold infinity of systems of three circles will be obtained, possessing the properties that one of the axes of similitude of all the systems and the centres of similitude lying in that axis are fixed, and that the lengths of the tangents to the circles from the fixed centres of similitude are invariable. The radical centre and the centres of the common tangent circles for all the systems will lie in a straight line perpendicular to the fixed axis of similitude. [To each axis of similitude correspond two tangent circles.]

(8) The locus of the centres of all the circles that touch a given circle and have a common axis of similitude is an ellipse.

Third Meeting, January 13th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

On Stirling's approximation to $n!$ when n is large.

By R. E. ALLARDICE, M.A.

The approximation is

$$n! = (n/e)^n \sqrt{2\pi n}.$$

We have

$$\left(\frac{n}{e}\right)^n = \left(\frac{n-1}{e}\right)^{n-1}.\frac{1}{e}.\left(\frac{n}{n-1}\right)^n.\left[n(n-1)\right]^{\frac{1}{2}}.\left(\frac{n-1}{n}\right)^{\frac{1}{2}}$$

$$= \left(\frac{n-1}{e}\right)^{n-1}.\frac{1}{e}.\left(\frac{n}{n-1}\right)^{n-\frac{1}{2}}.\left[n(n-1)\right]^{\frac{1}{2}}.$$

Now assume

$$\left(\frac{n}{n-1}\right)^{n-\frac{1}{2}} = e^x$$

$$\therefore\ x = (n-\tfrac{1}{2})\log\frac{n}{n-1}$$

$$= (-n+\tfrac{1}{2})\log\left(1-\frac{1}{n}\right)$$

$$= (-n+\tfrac{1}{2})\left(-\frac{1}{n}-\frac{1}{2}.\frac{1}{n^2}-\frac{1}{3}.\frac{1}{n^3}-\ \dots\ \dots\right)$$

$$= 1+\frac{1}{12}.\frac{1}{n^2}+\frac{B}{n^3}+\ \dots\ \dots$$

$$= 1+\frac{A}{n^2}$$

where A differs from $1/12$ by a quantity of the order of $1/n$, since the above series for x is convergent; and since n is large, A may be taken to be $1/12$.

Hence, $\left(\frac{n}{e}\right)^n = \left(\frac{n-1}{e}\right)^{n-1}.\frac{1}{e}.e^{1+A/n^2}.\left[n(n-1)\right]^{\frac{1}{2}}$

$$= \left(\frac{n-1}{e}\right)^{n-1}.e^{A/n^2}.\left[n(n-1)\right]^{\frac{1}{2}}.$$

Substituting for n in succession the values, $2n$, $2n-1$ ... ... $n+1$, we get

$$\left(\frac{2n}{e}\right)^{2n}=\Big[2n(2n-1)\ \dots\ \dots\ (n+1)(2n-1)\ \dots\ \dots\ n\Big]^{\frac{1}{2}}.\left(\frac{n}{e}\right)^{n}.e^{k}$$

where $$k=\frac{A}{(2n)^2}+\frac{A}{(2n-1)^2}+\ \dots\ \dots\ +\frac{A}{(n+1)^2}$$

$$\therefore\qquad \sqrt{2}\left(\frac{2n}{e}\right)^{2n}=\left(\frac{n}{e}\right)^{n}.\frac{(2n)\,!}{n\,!}.e^{k}.$$

$$\therefore\qquad \sqrt{2}\left(\frac{n}{e}\right)^{n}.2^{2n}=\frac{(2n)!}{n\,!}.e^{k}$$

$$\therefore\qquad n\,!=\left(\frac{n}{e}\right)^{n}.\sqrt{2}.2^{2n}.\frac{(n\,!)^2}{(2n)\,!}.e^{-k}.$$

Now, Wallis's formula for π is

$$\frac{\pi}{2}=\left[\frac{2.4.6\ \dots\ \dots\ 2n}{1.3.5\ \dots\ \dots\ 2n-1}\right]^{2}.\frac{1}{2n+1}$$

or $$\frac{\pi}{2}=\left[\frac{2.4.6\ \dots\ \dots\ 2n}{1.3.5\ \dots\ \dots\ 2n-1}\right]^{2}.\frac{1}{2n}.\left(1+\frac{1}{2n}\right)^{-1}$$

$$\therefore\qquad \sqrt{\pi}=\frac{2.4.6\ \dots\ \dots\ 2n}{1.3.5\ \dots\ \dots\ 2n-1}.\frac{1}{\sqrt{n}}.\left(1+\frac{1}{2n}\right)^{-\frac{1}{2}}$$

Hence we may write

$$\sqrt{\pi}=\frac{2^{2n}(n\,!)^2}{(2n)\,!}.\frac{1}{\sqrt{n}}.\left(1-\frac{1}{4n}\right).$$

This result reduces the above equation for $n\,!$ to

$$n\,!=\left(\frac{n}{e}\right)^{n}.\sqrt{2}.\sqrt{\pi n}.e^{-k}.\left(1+\frac{1}{4n}\right)$$

Now, k lies between $nA/(2n)^2$ and $nA/(n+1)^2$. Hence if we write $n\,!=\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}$, the ratio of the error to the value found for $n\,!$ is of the order of $1/n$.

It may now be shown that in the above approximation to π, the error made bears to π a ratio of the order of $1/n$.

In the value assumed for π

$$\frac{\pi}{2}=\left[\frac{2.4.6\ \dots\ \dots\ 2n}{1.3.5\ \dots\ \dots\ 2n-1}\right]^{2}.\frac{1}{2n+1}$$

the factor u has been neglected, where

$$u=\frac{(2n+2)^2}{(2n+1)(2n+3)}.\frac{(2n+4)^2}{(2n+3)(2n+5)}\ \dots\ \dots.$$

Now, $$\frac{(2n+2)^2}{(2n+1)(2n+3)}=\frac{4n^2+8n+4}{4n^2+8n+3}=1+\frac{1}{(2n+1)(2n+3)}$$

$$\therefore \qquad u=\left[1+\frac{1}{(2n+1)(2n+3)}\right]\left[1+\frac{1}{(2n+3)(2n+5)}\right] \;\ldots\; \ldots .$$

$$\therefore \qquad \log u=\frac{1}{(2n+1)(2n+3)}+\frac{1}{(2n+3)(2n+5)}+\;\ldots\;\ldots$$

$$-\tfrac{1}{2}.\left[\frac{1}{(2n+1)(2n+3)}\right]^2-\tfrac{1}{2}\left[\frac{1}{(2n+3)(2n+5)}\right]^2-.$$

Now $$\frac{1}{(2n+1)(2n+3)}=\tfrac{1}{2}\left[\frac{1}{2n+1}-\frac{1}{2n+3}\right].$$

Hence, neglecting small quantities of a higher order than $1/n$, we may write

$$\log u = 1/2(2n+1)$$

$$\therefore \qquad u = e^{1/2(2n+1)}$$

$$= 1 + 1/2(2n+1) + \;\ldots\;\ldots;$$

which shows that the error made in the above approximation to π is of the order stated.

[Another elementary proof of Stirling's theorem, by Mr J. W. L. Glaisher, is given in the *Quarterly Journal of Mathematics*, vol. xv., p. 57.]

A Device for the Analysis of Intervals and Chords in Music.

By A. Y. Fraser, M.A., F.R.S.E.

§ 1. The device here described has been found to simplify greatly the "somewhat laborious discussion" of the different musical intervals as given, say, in Sedley Taylor's *Sound and Music* (chap. viii.) or in Helmholtz's *Sensations of Tone*. It has been found particularly helpful in giving an account, necessarily rapid, of the nature of harmony to classes studying sound as a part of physics, from whom much familiarity with musical terms and notation is not to be expected.

§ 2. The leading idea of the device is that the octave being what may be called a periodic phenomenon, should be represented on a circular or spiral curve, and not, as is usual, on a straight vertical line. Such a representation of four octaves is given in figure 14. (To read this and the other figures, begin at the extreme right and

follow the spiral round counter-clock-wise.) The compactness of this representation as compared with that on a straight line is at once obvious. The different stretches of intervals are represented by the different sizes of angles (which are, as shown, proportional to the numbers 45, 40, and 24). The various observations that have to be made regarding intervals, for example, that the fourth is the inversion of the fifth, that the interval D to *m* is equal to the interval *s* to *t*, and so on, can be got from this figure, at least as well as from the usual vertical scale. It remains to show how this diagram can be developed to exhibit readily the nature of harmonies, that is, of two or more notes sounded together; but before this can be shown, some further preliminary explanation is necessary.

§ 3. When two *simple* tones are sounded together, the effect is agreeable* or the reverse, according to the interval between them. The effect is found to be unpleasant if the interval be less than a minor third; the interval of a tone being less disagreeable than the interval of a semi-tone, and any disagreeable interval becoming less disagreeable if both notes are taken at a higher pitch. This unpleasantness has been shown to be due to rapid waxings and wanings of sound, called *beats*, as the vibrations of the two notes assist or oppose each other. The interval of a minor third is regarded, in fact, as the smallest smooth interval. Such an interval can be readily distinguished on our diagram, for it is represented by what is almost exactly a right angle.

§ 4. So much for *simple* tones; but, if we except the tuning fork, which, in certain circumstances does so, no musical instrument gives out simple tones; every tone is *compound*—is, in fact, a congeries of simple tones. Such a compound tone is made up as follows: There is (A) a loud sounding simple tone and (B) a collection of simple tones higher in pitch than (A), differing in relative intensity, and all in general much weaker than the fundamental tone (A). These

---

* I use the words agreeable and disagreeable to save circumlocution, knowing all the time that musicians object *in toto* to the use of the words. A "disagreeable" interval may be used in a musical composition like a "disagreeable" incident in any other work of art, with the most happy general effect. Absolute smoothness in music, like absolutely pure water, is neither quite attainable, nor at all pleasing to the taste if attained.

component simple tones are called *partial tones*, the fundamental tone (A) being called the *prime partial*, and the rest (B) the *upper partials*. If the frequency of vibration of the prime partial be 1, then the series of upper partials have frequencies 2, 3, 4, 5, 6, &c., and thus if C represent the pitch of the prime partial tone, the whole series is (Fig. 15)

1,	2,	3,	4,	5,	6, &c.
C	C′	G′	C″	E″	G″, &c.

As a rule, beyond the last note given above, the upper partials are so weak as to be practically non-existent. The *quality* of any tone is determined by the number and relative intensities of the partial tones composing it, while the *pitch* of a tone is that of its prime partial.

§ 5. The last two sections are, of course, an epitome of facts, due for the most part to Helmholtz, and quite well known, but necessary for the understanding of the contrivance now to be described. In our examples we shall suppose the tones to be each made up of the six partials given above. They are the most important*, and the analysis of an interval with any other combination of partial tones is in principle quite the same.

§ 6. Figure 15 represents a diagram drawn on a large sheet of cardboard. Behind the cardboard is glued a strip of wood, which affords a support to a pin (say two inches of an ordinary pen holder), which projects from the centre of the spiral at right angles to the surface of the cardboard. This diagram, in which the six arrowheads represent six partials, will be taken to stand for all the partial tones of the lower of two notes whose combined effect is to be analysed.

Figure 16 represents a moveable arrangement to be used in connexion with figure 15. In the middle is a circle of wood with a hole in its centre, so that it can be made to rotate stiffly about the pin in fig. 15. This circle of wood carries radially three rods (steel knitting wires), and on each rod there slides a strip of cardboard cut out to represent one, two, and three arrowheads. The size of the arrangement is such that the whole can be made to cover exactly the corres-

* "The sounds of most musical instruments practically contain only the first six partial tones."—Sedley Taylor, *Sound and Music*, 2nd edit., p. 167.

ponding parts of fig. 15. Figure 16 is to be employed to represent the higher of two notes that are sounded together.

§ 7. Suppose, now, the framework of fig. 16 to be superposed on fig. 15 so as to coincide with it exactly. This will represent *unison*, that is, the sounding together of two notes of the same pitch and quality, with, therefore, complete coincidence in all the partial tones. Turn now the upper part a degree or two in either direction, and this will, of course, put every arrowhead of the one a degree or two from the corresponding arrowhead of the other. The interpretation of the diagram now is that there are six beating pairs of partials, which shows the *definiteness* of the unison combination—a chord being definite when it is bounded on each side by sharp discords.

§ 8. Again, let us look at the nature of the *major third* interval. Turn* round the moving part of our diagram till its "C" coincide with the "E" of the diagram on fig. 15. The dotted lines on fig. 17 show the position taken up by the moving part.

An examination of the diagram (fig. 17) shows that there is beating between the 4th partial of C and the 3rd of E, which two are a semi-tone apart, and also between the 6th of C and the 5th of E, also a semi-tone apart, but at a much higher pitch, and on that account, as well as from the fact that they will in general be weaker, they are a less rough pair. So much for the beating, due to partials in the major third.

If, now, the upper of the two notes be raised an octave (or the lower depressed an octave)—that is, if a male and a female voice sing the notes—the above beating intervals are cut out, and thus, so far as the first six partials are concerned, the *major tenth* (for such the interval now is) is a smooth interval. This bears out the known fact that a "major third" is smoother when taken by a male and a female voice than when taken by two females.

§ 9. We need not go into further detail in this direction. By

---

* As the moving part turns round to the left, its arrowheads project beyond the lines of the spiral to which they belong, and have either to be pushed back by hand or to be pulled back by strings passing round the fixed pin in the centre and shortening by being wound up. A small piece of india-rubber cord joining the point of each of the three cardboard strips of the outer extremity of its rod, will keep the string tight.

turning round the movable part (fig. 16, that is to say) every combination of two notes can be exhibited; angles less than a right angle—that is, intervals less than a minor third—at once catch the eye, and the numbers written on the arrowheads tell without trouble what partials are concerned. The roughness of any interval being thus ascertained, we can examine into the definiteness (due really to coinciding partials) by turning the upper part a degree or two out of position.

§ 10. One or two miscellaneous suggestions may be useful.

By using two moving parts like fig. 16 in conjunction with fig. 15, combinations of three notes can be studied. This requires no further illustration here.

A diagram such as fig. 14, with a spiral making four or five circuits, can be used as a repository of a good deal of information. One circuit may have the notes named as D, *r*, *m*, &c., while the next bears the letters C, D, E, &c. A third circle might carry the vibration ratios 1, $\frac{9}{8}$, $\frac{5}{4}$, &c., while the fourth might give a standard series of vibration numbers. The outermost circuit of a fair sized spiral would afford abundance of room for particulars concerning the various sharpened and flattened notes.

Our analysis has left combinational tones out of account, but the insertion of vibration numbers, as has just been suggested—even if a special diagram were devoted to this alone—would make the placing of the combinational tones an easy enough affair.

The method of using the diagram as described above is not the only way of turning the idea to account. A private student may use in connection with fig. 15 semi-transparent ground glass, which can be written on, or common glass with radial paper lines gummed on, or even a copy of fig. 16 cut out in cardboard; while for demonstration to a large audience the diagrams done on glass can be projected in the usual way by the lantern.

Fourth Meeting, February 10th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

On the inequality
$$mx^{m-1}(x-1) \gtreqless x^m - 1 \gtreqless m(x-1)$$
and its consequences.

By PROFESSOR CHRYSTAL.

§ 1. The object of this note is to establish the above inequality in as general a form as possible, and to prove by means of it two of the principal propositions in the theory of inequalities, one of which is usually proved by means of infinite series. The logical advantage in making the theory of inequalities independent of that of infinite series is obvious, when it is remarked that the discussion of the convergency of infinite series is strictly speaking a part of the theory of inequalities.

§ 2. *If x, p, q, are all positive, and p and q are integers, then $(x^p-1)/p > < (x^q-1)/q$ according as $p > < q$.*

Since p and q are integers
$$(x^p-1)/p > < (x^q-1)/q,$$
as
$$q(x^p-1) > < p(x^q-1),$$
as
$$(x-1)\{q(x^{p-1}+x^{p-2}+\ \dots\ +x+1) - p(x^{q-1}+x^{q-2}+\ \dots\ +x+1)\} > < 0.$$

Suppose $p > q$; and denote the expression on the left side of the last inequality by X. Then
$$X = (x-1)\{q(x^{p-1}+x^{p-2}+\ \dots\ +x^q) - (p-q)(x^{q-1}+x^{q-2}+\ \dots\ +x+1)\}.$$

Now, if $x > 1$, $\quad x^{p-1}+x^{p-2}+\ \dots\ +x^q > (p-q)x^q$,

and $\quad x^{q-1}+x^{q-2}+\ \dots\ +1 < qx^{q-1}$;

and therefore
$$\begin{aligned} X &> (x-1)\{q(p-q)x^q - (p-q)qx^{q-1}\} \\ &> q(p-q)x^{q-1}(x-1)^2 \\ &> 0. \end{aligned}$$

Again, if $x < 1$,
$$\begin{aligned} X &> (x-1)\{q(p-q)x^{p-1} - (p-q)q\} \\ &> q(p-q)(x-1)(x^{p-1}-1) \\ &> 0. \end{aligned}$$

Hence, in both cases, $(x^p-1)/p>(x^q-1)/q$.

By the same reasoning, if $q>p$,

$$(x^q-1)/q>(x^p-1)/p;$$

that is, if $p<q$, $\quad (x^p-1)/p<(x^q-1)/q$.

§ 3. If x is positive, and different from unity, then

$$mx^{m-1}(x-1)>x^m-1>m(x-1)$$

unless m lie between 0 and 1, in which case

$$mx^{m-1}(x-1)<x^m-1<m(x-1).$$

From last paragraph we have

$$(\xi^p-1)><(p/q)(\xi^q-1) \tag{1}$$

according as $p><q$; where ξ is any positive quantity, different from unity, and p and q are positive integers. In (1) we may put $x^{1/p}$ for ξ, where x is any positive quantity different from unity, the real positive value of the q^{th} root being taken; and we may put m for p/q, where m is any positive commensurable quantity.

The inequality then becomes

$$x^m-1><m(x-1) \tag{2}$$

according as $m><1$; which is part of the theorem.

In (2) put $1/x$ for x, then

$$(1/x)^m-1><m(1/x-1)$$

or

$$mx^{m-1}(x-1)><x^m-1. \tag{3}$$

according as $\quad m><1.$

The theorem is thus established for positive values of m.

Next, let $m=-n$, then

$$x^{-n}-1><(-n)(x-1),$$

according as $\quad 1-x^n><-nx^n(x-1),$

according as $\quad x^n-1><nx^n(x-1),$

according as $\quad nx^{n+1}-nx^n><x^n-1,$

according as $\quad (n+1)x^n(x-1)><x^{n+1}-1.$

Now, since n is positive, $n+1>1$; therefore by (3)

$$(n+1)x^n(x-1)>x^{n+1}-1;$$

and therefore

$$x^{-n}-1>(-n)(x-1). \tag{4}$$

In (4) write $1/x$ for x, then

$$(1/x)^{-n}-1>(-n)(1/x-1)$$

therefore $\quad x^{-n}-1<(-n)x^{-n-1}(x-1)$

that is, $\quad (-n)x^{-n-1}(x-1)>x^{-n}-1.$

Hence, if m is negative,

$$mx^{m-1}(x-1)>x^m-1>m(x-1);$$

which completes the demonstration.

§ 4. *The arithmetic mean of n positive quantities is not less than the geometric mean.*

Let us suppose the theorem to hold for n quantities $a, b, c, \ldots \ldots k$; and let l be one more.

By hypothesis, $(a+b+c+ \ldots \ldots +k)/n \nless (abc \ldots \ldots k)^{1/n}$

that is, $\quad a+b+c+ \ldots \ldots +k \nless n(abc \ldots \ldots k)^{1/n}$

Therefore $\quad a+b+c+ \ldots \ldots +k+l \nless n(abc \ldots \ldots k)^{1/n}+l.$

Now, $\quad n(abc \ldots \ldots k)^{1/n}+l \nless (n+1)(abc \ldots \ldots kl)^{1/(n+1)},$

provided $\quad n(abc \ldots \ldots k/l^n)^{1/n}+1 \nless (n+1)(abc \ldots \ldots kl/l^{n+1})^{1/(n+1)},$

$$\nless (n+1)(abc \ldots \ldots k/l^n)^{1/(n+1)};$$

that is, provided $\quad n\xi^{n+1}+1 \nless (n+1)\xi^n,$

where $\quad \xi^{n(n+1)} = abc \ldots \ldots k/l^n,$

that is, provided $\quad (n+1)\xi^n(\xi-1) \nless \xi^{n+1}-1,$

which is true, by the theorem of last paragraph.

Hence, if the theorem hold for n quantities, it will hold for $n+1$; and it is obviously true for two quantities, and hence it is true generally.

Corollary.—If $a, b, \ldots \ldots k$ are n positive quantities, and $p, q, \ldots \ldots t$, n positive commensurable quantities, then

$$\frac{pa+qb+ \ldots \ldots +tk}{p+q+ \ldots \ldots +t} > (a^p b^q \ldots \ldots k^t)^{1/(p+q+ \ldots \ldots +t)}$$

It is obvious that we are only concerned with the ratios of the quantities $p, q, \ldots \ldots t$; and we may, therefore, suppose these quantities to be integral. The theorem is thus seen to be a particular case of that just proved—namely, that the arithmetic mean of $p+q+ \ldots \ldots t$ positive quantities, of which p are equal to a, q to b, and so on, is greater than their geometric mean.

§ 5. *If $a, b, \ldots \ldots k$, are n positive quantities, and $p, q, \ldots \ldots t$, are n commensurable quantities, then*

$$\frac{pa^m+qb^m+ \ldots \ldots +tk^m}{p+q+ \ldots \ldots +t} \nless \ngtr \left(\frac{pa+qb+ \ldots \ldots +tk}{p+q+ \ldots \ldots +t}\right)^m \qquad (1)$$

according as m does not or does lie between 0 and 1.

If we denote $p(p+q+ \ldots \ldots +t)$, $q(p+q+ \ldots \ldots +t)$, etc., by $\lambda, \mu, \ldots \ldots \tau$; and $a/(\lambda a+\mu b+ \ldots \ldots +\tau k)$, $b/(\lambda a+\mu b+ \ldots \ldots +\tau k)$, etc., by $x, y, \ldots \ldots w$, then

$$\lambda+\mu+ \ldots \ldots +\tau=1$$

$$\lambda x+\mu y+ \ldots \ldots +\tau w=1.$$

Dividing both sides of (1) by $\{(pa+qb+ \ldots tk)/(p+q+ \ldots +t)\}^m$ we have to prove that

$$\lambda x^m + \mu y^m + \ \dots \ \dots \ + \tau w^m \nless \ngtr 1$$

according as m does not or does lie between 0 and $+1$.

Now, if m does not lie between 0 and 1,

$$x^m - 1 \nless m(x-1),\ y^m - 1 \nless m(y-1), \text{ etc.}$$

Therefore
$$\begin{aligned}\Sigma\lambda(x^m - 1) &\nless \Sigma\lambda m(x-1)\\ &\nless m\{\Sigma\lambda x - \lambda\}\\ &\nless m(1-1)\\ &\nless 0;\end{aligned}$$

that is,
$$\Sigma\lambda x^m \nless \Sigma\lambda \nless 1.$$

In like we may show that if m does lie between 0 and 1, then

$$\Sigma\lambda x^m \ngtr 1.$$

Corollary.—If we make $p = q = \ \dots \ \dots \ = t$, we have

$$(a^m + b^m + \ \dots \ \dots \ + k^m)/n \nless \ngtr \{(a + b + \ \dots \ \dots \ + k)/n\}^m,$$

that is to say, *the arithmetical mean of the m^{th} powers of n positive quantities is not less or not greater than the m^{th} power of their arithmetical mean, according as m does not or does lie between* 0 *and* 1.

§ 6. The inequality just discussed is by no means new, nor has its importance been overlooked, as may be seen from the elegant use of it by Schlömilch in the second chapter of his *Algebraische Analysis.* (See also *Zeitschrift für Mathematik*, Bd. III., p. 387 (1858), and Bd. VII., p. 46 (1862); also G. F. Walker, *Messenger of Mathematics*, vol. XII., p. 37). The inequality has not, however, usually been stated in quite so general a form as the one I have given; and, possibly in consequence, its application to the demonstration of the theorem of § 5 seems to have hitherto escaped notice. This theorem is usually proved by a somewhat awkward combination of induction and the use of infinite series.

The history of the theorem is a little obscure. At first I suspected that it was due to Cauchy, but it does not appear in his *Analyse Algébrique* (Paris, 1821). The earliest reference to it which I have discovered was given me by Mr A. Y. Fraser, and occurs in *Problèmes et Dévelopmens sur diverses Parties des Mathématiques*, par M. Reynaud et M. Duhamel. It is there deduced from the maximum and minimum values of $x^m + y^m$ subject to the condition $ax + by = c$. I can scarcely believe that this is the earliest occurrence, and I should be glad if any of our members could furnish me with reference to an earlier.

§ 7. The inequality $mx^{m-1}(x-1) \gtreqless x^m - 1 \gtreqless m(x-1)$ has the merit

of binding together a great variety of algebraical theorems which are usually put before the student without any organic connection whatever; and for this reason I have brought it specially under the notice of the younger members of the Mathematical Society. Its power is not surprising when we reflect on its close connection with the theorem $\underset{x=1}{\mathrm{L}}(x^m - 1)/(x - 1) = m$, which is the fundamental proposition in the differentiation of algebraic functions.

Mr W. PEDDIE exhibited and described a model of the thermodynamic surface which represents the state of water-substance in terms of pressure, volume, and temperature. Various lines, the equations of which are $\frac{dp}{dt} = \text{const.}$, $\frac{dp}{dv} = \text{const.}$, &c., were drawn upon the surface.

Fifth Meeting, March 9th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

Sur un système de cercles tangents à une circonférence et orthogonaux à une autre circonférence.

Par M. PAUL AUBERT.

On donne deux cercles S et Σ ayant pour centres les points O et ω, pour rayons r et ρ. Le cercle S est supposé intérieur au cercle Σ, et le point ω intérieur au cercle S.

I. *Tous les cercles T tangents extérieurement au cercle S et ortho gonaux au cercle Σ sont tangents à un troisième cercle fixe.*

Figures 18, 19.

Soit T un cercle tangent au cercle S et orthogonal à Σ. Prenons la figure inverse par rapport au point I comme pôle, la puissance d'inversion étant la puissance k^2 du point I par rapport au cercle S. Ce cercle reste invariable, et le cercle Σ se transforme en une droite perpendiculaire au diamètre Iω en un point P′ tel que

$$\mathrm{IP}\cdot\mathrm{IP}' = k^2.$$

Le cercle T se transforme en un cercle T′ tangent au cercle S et coupant à angle droit la droite P′D; son centre est donc sur cette droite, et par suite le cercle T′ est aussi tangent à la circonférence S_1

symétrique de S par rapport à la droite P'D. Il en résulte que le cercle T lui-même est tangent à la circonférence U qu'on obtiendrait en transformant S_1, le pôle étant au point I, et la puissance d'inversion égale à k^2. La circonférence ainsi obtenue étant fixe quand on considère tous les cercles tels que T, la proposition est démontrée.

Cherchons son centre V et son rayon.*

Désignons par d la distance $O\omega$. On sait que les circonférences U et S_1 sont homothétiques par rapport au pôle I, et que le rapport de similitude est égal au quotient de la puissance d'inversion par la puissance du pôle relative au cercle S_1.

Ainsi $$\frac{IV}{IO_1}=\frac{\epsilon k^2}{\overline{IO_1}^2-r^2} \qquad \text{où } \epsilon^2=1.$$

En remarquant que
$$IO+IO_1=2IP'=k^2/\rho,$$
puis substituant à IO sa valeur $(d+\rho)$,
et à k^2 l'expression $[(d+\rho)^2-r^2]$,

il vient $$IO_1=\frac{\epsilon[(d+\rho)d-r^2]}{\rho}$$
$$IV=\frac{\epsilon\rho}{r^2-d^2}[d(d+\rho)-r^2].$$

Son rayon est alors $$R=\frac{\rho^2 r}{r^2-d^2}.$$

On obtient d'ailleurs immédiatement le point V et le rayon R par une construction géométrique. Ayant en effet démontré l'existence du cercle U tangent à la circonférence T, désignons par B le point de contact, et par A le point où la circonférence T touche le cercle S. Si nous joignons BA, les points B et A étant deux points antihomologues des circonférences S et U, considérées comme inversement homothétiques, la droite BA va couper la ligne des centres VO au centre d'homothétie inverse de ces deux circonférences. Je dis que ce point n'est autre que ω.

En effet, désignons le pour un instant par C. Le produit $CA\times CB$ est constant pour tous les cercles tels que T tangents à S et à U ; soit h^2 sa valeur. Tous les cercles T seront donc orthogonaux à un cercle fixe ayant C pour centre et dont le rayon est h. Mais nous savons déjà que ces cercles T sont orthogonaux au cercle Σ.

* La solution est un peu plus simple si l'on prend pour pôle d'inversion le centre ω du cercle Σ, et pour uissance d'inversion le carré de son rayon.

Les deux cercles O et Σ coïncident donc, sans quoi le lieu des centres des circonférences T serait l'axe radical des cercles O et Σ, ce qu'il est absurde de supposer puisque tous ces cercles sont tangents aux circonférences S et U. Donc la droite BA passe par le point ω, qui est le centre d'homothétie inverse des deux circonférences S et U.

Cela posé, pour obtenir le point V, on construira un cercle T quelconque par la méthode connue, puis on joindra ωA qu'on prolongera jusqu' en B. La droite BT coupera le diamètre ωI au point V cherché ; le rayon de la circonférence est BV.

II. *On prend une droite quelconque LL' perpendiculaire à la ligne des centres Oω. Soient ωF, ωG les tangentes menées du point ω à l'un des cercles T ; soient F' et G' les points d'intersection de la droite LL' avec les bissectrices des angles FωO et GωO. Les points F' et G' forment une division homographique quand le cercle T varie.*

Posons $F'L = x$, $G'L = y$; il faut montrer que x et y satisfont à une relation de la forme

$$mxy + nx + py + q = 0.$$

La figure nous donne

$$GT = \rho \operatorname{tg} \frac{G\omega F}{2}.$$

Or,

$$\frac{G\omega F}{2} = \frac{G\omega L}{2} - \frac{F\omega L}{2},$$

et

$$\operatorname{tg} \frac{G\omega L}{2} = \frac{y}{\omega L}, \quad \operatorname{tg} \frac{F\omega L}{2} = \frac{x}{\omega L};$$

d'où, en posant $\omega L = l$,

$$(1) \qquad GT = \rho \frac{l(y - x)}{l^2 + xy}.$$

Cherchons une autre expression de GT. On a

$$(2) \qquad \overline{OT}^2 = \overline{\omega O}^2 + \overline{\omega T}^2 - 2\omega O \cdot \omega T \cos O\omega T.$$

D'ailleurs on sait que

$$\overline{OT}^2 = (r + GT)^2, \quad \overline{\omega O}^2 = d^2, \quad \overline{\omega T}^2 = \overline{GT}^2 + \rho^2,$$

et

$$\cos O\omega T = \cos\left(\frac{G\omega L}{2} + \frac{F\omega L}{2}\right),$$

$$= \cos\frac{G\omega L}{2}\cos\frac{F\omega L}{2} - \sin\frac{G\omega L}{2}\sin\frac{F\omega L}{2}.$$

Remplaçons ces cosinus et sinus par leurs expressions au moyen des tangentes des mêmes arcs, il vient

$$\cos O\omega T = \frac{l^2 - xy}{\sqrt{(x^2 + l^2)(y^2 + l^2)}},$$

et la formule (2) devient

$$(r+\mathrm{GT})^2 = d^2+\rho^2+\overline{\mathrm{GT}}^2 - 2d\sqrt{\rho^2+\overline{\mathrm{GT}}^2}\frac{l^2-xy}{\sqrt{(x^2+l^2)(y^2+l^2)}}$$

c'est à dire

$$r^2+2r\cdot\mathrm{GT} = d^2+\rho^2-2d(l^2-xy)\frac{\sqrt{\overline{\mathrm{GT}}^2+\rho^2}}{\sqrt{(x^2+l^2)(y^2+l^2)}}.$$

Or, en tenant compte de la relation (1), on a

$$\overline{\mathrm{GT}}^2+\rho^2 = \rho^2\frac{l^2(y-x)^2+(l^2+xy)^2}{(l^2+xy)^2},$$

ou

$$\overline{\mathrm{GT}}^2+\rho^2 = \frac{\rho^2}{(l^2+xy)^2}[l^4+(x^2+y^2)l^2+x^2y^2],$$

$$\overline{\mathrm{GT}}^2+\rho^2 = \frac{\rho^2}{(l^2+xy)^2}(l^2+x^2)(l^2+y^2).$$

Par suite on peut écrire

$$r^2+2r\cdot\mathrm{GT} = d^2+\rho^2-2d\rho\frac{l^2-xy}{l^2+xy}.$$

Si nous égalons l'expression (1) de GT avec celle que fournit cette dernière relation, il vient, après quelques simplifications

(3) $$[(\rho+d)^2-r^2]xy+2lr\rho(y-x)+l^2[(\rho-d)^2-r^2]=0.$$

C'est bien une relation d'homographie entre x et y.

On peut, en remarquant qu' x et y n'y figurent que par les rapports x/l et y/l, ce qu'on pouvait prévoir, poser

$$x/l = x_1,\ y/l = x_2,$$

avec $$\mathrm{A}=(\rho+d)^2-r^2,\ \mathrm{B}=r\rho,\ \mathrm{C}=(\rho-d)^2-r^2,$$

et on a la relation

(4) $$\mathrm{A}x_1x_2+2\mathrm{B}(x_2-x_1)+\mathrm{C}=0.$$

III. *Soient T_1, T_2, T_3, ... T_n une série de cercles T orthogonaux au cercle Σ, le premier aux points A_1 et A_2, le second aux points A_2 et A_3, le troisième aux points A_3 et A_4, ... le $n^{\text{ième}}$ aux points A_n et A_{n+1}. La condition nécessaire et suffisante pour que le point A_{n+1} coïncide avec le point A_1 est que l'on puisse satisfaire à une relation de la forme*

$$\mathrm{tg}\frac{k\pi}{n} = \frac{1}{2r\rho}\sqrt{(\rho^2+d^2-r^2)^2-4d^2\rho^2}.$$

Désignons d'une manière générale par son ordonnée x_p le point de rencontre de la bissectrice de l'angle $\mathrm{A}_p\omega\mathrm{L}$ avec la perpendiculaire LL'. Il est clair que si le point A_{n+1} vient coïncider avec le point A_1, le point x_{n+1} coïncidera avec le point x_1; et réciproquement.

Nous sommes donc ramené à chercher la condition nécessaire et suffisante pour que la valeur de x_{n+1} obtenue en appliquant successivement la formule (4) à toutes les valeurs consécutives des indices jusqu' à l'indice $(n+1)$, soit égale à la valeur primitive x_1. Nous tirons de (4)

$$x_2 = \frac{2Bx_1 - C}{Ax_1 + 2B}.$$

Désignons par a et b les racines de l'équation du second degré en x obtenue en supposant dans la relation (4) $x_1 = x_2 = x$. On a

$$a + b = 0, \quad Aab = C.$$

Substituons à C cette valeur, il vient

$$x_2 = \frac{2Bx_1 - Aab}{Ax_1 + 2B},$$

d'où
$$\frac{x_2 - a}{x_2 - b} = \frac{2Bx_1 - Aab - Aax_1 - 2Ba}{2Bx_1 - Aab - Abx_1 - 2Bb},$$

ou
$$\frac{x_2 - a}{x_2 - b} = \frac{(2B - Aa)(x_1 - a) - Aa(a+b)}{(2B - Ab)(x_1 - b) - Bb(a+b)}.$$

Mais $a + b = 0$; donc

$$\frac{x_2 - a}{x_2 - b} = \frac{2B - Aa}{2B - Ab} \cdot \frac{x_1 - a}{x_1 - b}.$$

On aura pareillement

$$\frac{x_3 - a}{x_3 - b} = \frac{2B - Aa}{2B - Ab} \cdot \frac{x_2 - a}{x_2 - b},$$
$$= \left(\frac{2B - Aa}{2B - Ab}\right)^2 \cdot \frac{x_1 - a}{x_1 - b},$$

et en général

$$\frac{x_{n+1} - a}{x_{n+1} - b} = \left(\frac{2B - Aa}{2B - Ab}\right)^n \cdot \frac{x_1 - a}{x_1 - b}.$$

Si on a $x_{n+1} = x_1$, cette relation donne

$$\left(\frac{2B - Aa}{2B - Ab}\right)^n = 1.$$

Par suite
$$\frac{2B - Aa}{2B - Ab} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}.$$

Mais, A et C étant dans nos hypothèses des quantités essentiellement positives, on a

$$a = -\sqrt{\frac{C}{A}}\,i, \quad b = +\sqrt{\frac{C}{A}}\,i,$$

d'où
$$Aa = -\sqrt{AC}\,i, \quad Ab = +\sqrt{AC}\,i;$$

et l'on a

$$\frac{2B + \sqrt{AC}\,i}{2B - \sqrt{AC}\,i} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}.$$

En égalant les parties réelles et les parties imaginaires, il vient

$$(5)\qquad 4B^2 - AC = (4B^2 + AC)\cos\frac{2k\pi}{n},$$

$$(6)\qquad 4B\sqrt{AC} = (4B^2 + AC)\sin\frac{2k\pi}{n},$$

d'où (7)
$$\operatorname{tg}\frac{2k\pi}{n} = \frac{4B\sqrt{AC}}{4B^2 - AC},$$

condition nécessaire et suffisante pour que les relations (5) et (6) soient satisfaites, car le module du premier membre de la relation précédente est égal à l'unité. On tire de (7)

$$\operatorname{tg}\frac{k\pi}{n} = \frac{4B^2 - AC \pm \sqrt{16ACB^2 + (4B^2 - AC)^2}}{4B\sqrt{AC}}$$

c'est à dire
$$\operatorname{tg}\frac{k\pi}{n} = \frac{4B^2 - AC \pm (4B^2 + AC)}{4B\sqrt{AC}}.$$

En remplaçant A, B, C par leurs valeurs, il vient

$$\operatorname{tg}\frac{k\pi}{n} = \frac{2r\rho}{\sqrt{(\rho^2 + d^2 - r^2)^2 - 4d^2\rho^2}}$$

et
$$\operatorname{tg}\frac{k\pi}{n} = \frac{-\sqrt{(\rho^2 + d^2 - r^2)^2 - 4d^2\rho^2}}{2r\rho}.$$

Ces deux valeurs sont inverses l'une de l'autre et de signes contraires, ce qu'on savait. On peut donc simplement conserver la dernière, et, en ne considérant que la valeur absolue, écrire

$$\operatorname{tg}\frac{k\pi}{n} = \frac{1}{2r\rho}\sqrt{(\rho^2 + d^2 - r^2)^2 - 4d^2\rho^2}.$$

C'est bien la relation donnée.

The Nine-Point Circle.

By Rev. JOHN WILSON, M.A.

I. To find the nine points.

(1) Let M (fig. 20) be the centre of the circle which passes through H, K, L, the middle points of BC, BA, CA.

Draw the diameter HMU.

Join UX, UA, and the three lines forming the median $\triangle$ HKL.

UX is $\perp$ to BC and KL.

(2) Let $\triangle$ HKL be turned through 180° round M, then H, K, L will assume the positions U, V, W;
and WU = and $\parallel$ HL is also = and $\parallel$ AK, the half of AB.

Hence AU is = and $\parallel$ KW.

(3) Now the $\angle$ LKW is a right angle.

Hence KW and AU are $\perp$ KL (and BC),
and therefore $\parallel$ UX.

AUX is therefore a straight line $\perp$ to BC.

Similarly BZ, and CY are straight lines, and they intersect in O, the orthocentre;

$$\left.\begin{array}{l}\text{and H, K, L}\\ \quad\text{U, V, W}\\ \quad\text{X, Y, Z}\end{array}\right\}\text{are points in the circle.}$$

H, K, L are the middle points of the $\triangle$ ABC;

X, Y, Z the feet of the $\perp^{ars}$ from the vertices A, C, B on the opposite sides;

U, V, W the points in which these $\perp^{ars}$ cut the circle.

II. To show that U, V, W, the points in which the $\perp^{ars}$ from the vertices to the opposite sides cut the medioscribed circle, bisect the segments of the $\perp^{ars}$ between the orthocentre and the vertices.

In the $\triangle$ AOB,
KW is drawn through the middle point of AB $\parallel$ to the base;
hence, KW $= \frac{1}{2}$AO.
$\therefore$ AU = UO.

Similarly BW = WO and CV = VO.

III. To show that S, the circum-centre, corresponds with O the ortho-centre. Suppose the $\triangle$ HKL swung round as before.

If LS and HS be drawn $\perp$ to the sides, S is the centre of the circumscribed circle.

In the triangles LHS, UWO

$$\left.\begin{array}{l}\text{LH is} \parallel \text{and} = \text{UW}\\ \text{HS} \parallel \text{UO}\\ \text{LS} \parallel \text{WO}\end{array}\right\}$$

Hence the triangles are congruent
and HS = UO.

IV. The line joining S and O is bisected in M.

Since HS is $\parallel$ and = UO,
UOHS forms a $\parallel^m$ whose diagonals mutually bisect.
But UH is bisected in M;
hence SO passes through and is bisected in M.

V. If the median AH be drawn it will cut OS in G so that OG = 2GS.

Through U draw UP $\parallel$ OG;
then UP = $\frac{1}{2}$OG.

AG is bisected in P, hence a line PQ drawn through P $\parallel$ AO bisects OG and is = $\frac{1}{2}$AO = UO = SH.

Hence PQHS is a $\parallel^m$ whose diagonals mutually bisect.

Hence QG = GS,

or GS = $\frac{1}{3}$OS.

HG is also equal to $\frac{1}{3}$AH.

Hence G is the point of intersection of the medians of the $\triangle$ ABC.

VI. O, M, G, and S, respectively the ortho-centre, the centre of the medioscribed circle, the centroid and the centre of the circumscribed circle, are collinear.

Sixth Meeting, April 13th, 1888.

W. J. Macdonald, Esq., M.A., F.R.S.E., President, in the Chair.

Extension of a theorem of Abel's in summation to integration.

By George A. Gibson, M.A.

The extension referred to was first given, I believe, by M. Ossian Bonnet in *Liouville's Journal*, vol. xiv., pp. 249 *et seq.* M. Jordan, in his *Cours d'Analyse*, tom. 2, § 82, seems to refer to this article in attributing to M. Bonnet the discovery of the "Second Theorem of the Mean," though he does not explicitly say so. In a course of lectures on "Simple and Multiple Integrals," delivered at Berlin University in the summer of 1885 by Professor Kronecker, which I attended, the "Second Theorem of the Mean" was shown to be a case of Abel's theorem, though I do not remember that Professor Kronecker mentioned M. Bonnet's name in connection with it. I

have therefore thought that it might not be without interest to the members of this society to direct attention to the theorem, and to M. Bonnet's connection with it. The proof given is very similar to that in Jordan's *Cours d'Analyse.*

The theorem of Abel in question is Theorem III. of the introduction to his memoir on the Binomial Series (*Collected Works*, vol. I., p. 222). For present purposes we may state the theorem as follows:—If $A_0, A_1, \ldots A_m$ bea series of quantities* such that for all values of r from 1 to m, $A_r - A_{r-1}$ retains the same sign, and if P_r (where $P_r = B_0 + B_1 + \ldots + B_r$, and B_0, B_1, &c., are any quantities whatever) always lies between M and N, then the sum

$$Q = A_0B_0 + A_1B_1 + \ldots + A_mB_m$$

will always lie between MA_0 and NA_0.

We have $P_0 = B_0, P_1 - P_0 = B_1 \ldots P_m - P_{m-1} = B_m$

$$\begin{aligned} \therefore Q &= A_0P_0 + A_1(P_1 - P_0) + \ldots + A_r(P_r - P_{r-1}) + \ldots + A_m(P_m - P_{m-1}) \\ &= (A_0 - A_1)P_0 + (A_1 - A_2)P_1 + \ldots + (A_{r-1} - A_r)P_r + \ldots \ldots \ldots \\ &\qquad + (A_{m-1} - A_m)P_{m-1} + A_mP_m. \end{aligned}$$

Suppose $M > P_r > N$ and we see that Q lies in value between MA_0 and NA_0 since $A_r - A_{r-1}$ is always of the same sign. We may therefore write $Q = [N + \theta(M - N)]A_0$, where θ is a positive proper fraction.

If we now suppose the *As* and *Bs* to be functions of a variable x, we may get some of the ordinary theorems of integration.

Let $A_r = f(x_r)$ where $f(x)$ always varies in the same sense from x to x_m, and let $B_r = \phi(x_r)h_r$ where h_r is infinitesimal.

$\therefore\ P_r = \int_{x_0}^{x_r} \phi(x)dx$ and M and N are the greatest and least values of $\int_{x_0}^{x_r} \phi(x)dx\ (r = 1, 2 \ldots m)$

$$\therefore\ Q = \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M - N)]f(x_0) = f(x_0)\int_{x_0}^{\xi}\phi(x)dx$$

where ξ lies between x_0 and x_m. [It is evident that $M + \theta(M - N)$ can be put in the form $\int_{x_0}^{\xi} \phi(x)dx$.]

---

* $A_0, A_1, \ldots \ldots A_m$ are supposed to be positive quantities.

This result is very like that given by M. Bonnet.

Again, we have

$$Q = \sum_{r=0}^{r=m} A_0B_0 = \sum_{r=1}^{r=m}(A_{r-1} - A_r)P_{r-1} + A_mP_m$$

and, as before, $\sum_{r=1}^{r=m} (A_{r-1} - A_r)P_{r-1}$ lies between $M(A_0 - A_m)$ and $N(A_0 - A_m)$

$$\therefore\ Q = [N + \theta(M - N)](A_0 - A_m) + A_mP_m$$

$$\therefore \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M - N)](f(x_0) - f(x_m)) + f(x_m)\int_{x_0}^{x_m}\phi(x)dx$$

$$= (f(x_0) - f(x_m))\int_{x_0}^{\xi}\phi(x)dx + f(x_m)\int_{x_0}^{x_m}\phi(x)dx$$

$$= f(x_0)\int_{x_0}^{\xi}\phi(x)dx + f(x_m)\int_{\xi}^{x_m}\phi(x)dx$$

which is the ordinary form of the "Second Theorem of the Mean."

Lastly, we may note that the theorem of "Integration by parts" is virtually given in Abel's theorem, for we have

$$Q = A_0P_0 + \sum_{r=1}^{r=m} A_r(P_r - P_{r-1}) = \sum_{r=1}^{r=m}(A_{r-1} - A_r)P_r + A_mP_m$$

Let $A_r = f(x_r)$ and $P_r = \phi(x_r)$

$$\therefore \qquad A_{r-1} - A_r = -\frac{df(x_r)}{dx}h_r, \quad P_r - P_{r-1} = \frac{d\phi(x_r)}{dx_r}h_r$$

$$\therefore \quad f(x_0)\phi(x_0) + \int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - \int_{x_0}^{x_m} f'(x)\phi(x)dx$$

i.e., $$\int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - f(x_0)\phi(x_0) - \int_{x_0}^{x_m} f'(x)\phi(x)dx.$$

On the inscription of a triangle of given shape in a given triangle.

By R. E. Allardice, M.A.

§ 1. *To inscribe in a triangle ABC a triangle similar to the triangle DEF, and having its sides parallel to those of DEF.*

In order to inscribe in the triangle ABC (fig. 21), a triangle

having its sides parallel to those of DEF, through D, E, F, draw lines parallel to the sides of ABC, and then reduce the figure A′B′C′ in the ratio BC : B′C′.

Thus an infinite number of triangles may be inscribed in a given triangle, similar to another given triangle.

A direct construction may also be given, as follows:

In BC (fig. 22) take any point G: draw GH parallel to DE; HK parallel to EF; KL parallel to FD. Then we may easily calculate the ratio A′L : A′G where A′ is the vertex of the required triangle that lies in BC.

For,
$$\frac{A'L}{A'G} = \frac{A'L}{C'K} \cdot \frac{C'K}{B'H} \cdot \frac{B'H}{A'G}$$
$$= \frac{LB}{BK} \cdot \frac{KA}{AH} \cdot \frac{HC}{CG} \text{ which is known.}$$

[Dr Mackay suggests a modification of this method, depending on the fact that A, A′ and the point of intersection of KL and HG are collinear.]

The theorems of §§ 2, 3, and 4 are required further on in this paper.

§ 2. *To find the condition that the perpendiculars to the sides of a triangle ABC, drawn at the points D, E, F, in the sides, be concurrent.*

Let the perpendiculars at D, E, F, (fig. 23) meet in the point O.

Since $AO^2 - BO^2 = AF^2 - BF^2$, the necessary and sufficient condition for concurrence is obviously
$$AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0$$
or,
$$AF^2 + BD^2 + CE^2 = BF^2 + CD^2 + AE^2.$$

§ 3. *If one triangle is inscribed in another triangle, and if the perpendiculars from the vertices of one triangle on the sides of the other triangle are concurrent, then the perpendiculars from the vertices of the second triangle on the sides of the first are also concurrent.*

Let DEF (fig. 24) be inscribed in ABC; and let the perpendiculars at D, E, F, to the sides BC, CA, AB, meet in the point O; then the perpendiculars from A, B, C, on the sides of EF, FD, DE, will also meet.

Let the perpendiculars from A, B, C, meet the sides of DEF in A′, B′, C′.

Since the perpendiculars at D, E, F, are concurrent,

$$\therefore \qquad AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0.$$

But $\qquad AF^2 - AE^2 = A'F^2 - A'E^2$, etc.

$$\therefore \qquad A'F^2 - B'F^2 + B'D^2 - C'D^2 + C'E^2 - A'E^2 = 0;$$

which proves the theorem.

§ 4. *The lines joining the points O and P of last paragraph to the vertices are equally inclined to the bisectors of the angles of the triangle.*

Let O and P (fig. 24) be the two points.

Then, $\angle FAO = \angle FEO =$ complement of $\angle FEA = \angle EAP$.

In the recent geometry of the triangle, lines equally inclined to the bisector of an angle are called *isogonal* lines; and the points of concurrence of one set of three lines passing through the vertices and of the three isogonal lines are called *inverse* points. Thus OA and PA, OB and PB, OC and PC, are pairs of isogonal lines; and O and P are inverse points.

§ 5. The problem that first suggested itself, and that led to this paper, was as follows:

To find a point P within a triangle such that the images Q, R, S, of P in the three sides shall be the vertices of an equilateral triangle.

This is obviously the same problem as the following:

To inscribe, in a given triangle, an equilateral triangle, such that the perpendiculars to the sides of the given triangle, drawn through the vertices of the equilateral triangle, shall be concurrent.

First Method.

This problem may be solved by finding the point inverse (in the sense mentioned above) to the point of intersection of the perpendiculars through the vertices of the required equilateral triangle.

The corresponding point is, in fact, the point of concurrence of the circles circumscribing the equilateral triangles, described on the sides of the given triangle.

Suppose DEF (fig. 24) to be equilateral; then

$$\angle PAC = \angle OAF = \angle OEF,$$
$$\angle PCA = \angle OCB = \angle OED;$$
$$\therefore \angle PAC + \angle PCA = \angle DEF = 60^\circ, \therefore \angle APC = 120^\circ.$$

Thus the problem is solved.

§ 6. Eight systems of three circles may be obtained as the circles

circumscribing equilateral triangles described on the sides of a given triangle. The question naturally arises, In how many of these systems do the circles concur in one real point? We shall show that this happens only in the case of two of the systems.

First Case. Consider a triangle ABC with the angle C greater than 120°.

There is obviously a point of concurrence on each of the arcs of 120° described on AB, and none on either of the arcs of 60° described on AB.

Second Case. Consider a triangle ABC, with no angle greater than 120°.

There is always one point of concurrence of circles within the triangle, and one without the triangle. If the triangle has only one angle greater than 60°, the point of concurrence that lies outside the triangle is on the arc of 120° described on the greatest side of the triangle; while if the triangle has two angles greater than 60°, this point of concurrence is on the arc of 120° described on the shortest side of the triangle.

We may also show analytically, by getting the equations to the circles, that in only two of the systems do the three circles meet in one point.

The equation to the circle circumscribing the equilateral triangle described *externally* on the side BC is

$$(a\beta\gamma + b\gamma\alpha + c\alpha\beta)\sin 60^\circ - \gamma(a\alpha + b\beta + c\gamma)\sin(60^\circ + \mathrm{A}) = 0\,; \qquad (1)$$

while the equation to the circle circumscribing the equilateral triangle described *internally* on the same side (that is, so that the remaining vertex of the equilateral triangle and the vertex A are on the same side of BC), is

$$(a\beta\gamma + b\gamma\alpha + c\alpha\beta)\sin 60^\circ - \gamma(a\alpha + b\beta + c\gamma)\sin(60^\circ - \mathrm{A}) = 0. \qquad (2)$$

Now, it may be shown that the circle (1) and the other two corresponding circles meet in a point, and that the same is true of the circle (2) and the two other circles corresponding to it; and that these are the only two systems containing three concurrent circles.

§ 7. Another construction may be given for the points P of § 6, that is, for the points of concurrence of the circles circumscribing the equilateral triangle described on the sides of the given triangle.

It may, in fact, easily be shown that if equilateral triangles BCD, CAE, ABF, (fig. 25) are described on the sides of the triangle ABC,

the vertices A and D being on opposite sides of BC, B and E on opposite sides of CA and C and F on opposite sides of AB, then the lines AD, BE, CF are concurrent; and that the point of concurrence is the point of concurrence of the circles circumscribing the triangles BCD, CAE, ABF.

The same result follows if the equilateral triangles are described so that A and D are on the same side of BC, B and E on the same side of CA and C and F on the same side of AB.

§ 8. Second Method.

Let ABC (fig. 24) be the given triangle, DEF the required inscribed equilateral triangle; then, by the formula for the chord of a circle, in terms of the angle it subtends at the circumference and the diameter of the circle,

$$\mathrm{EF} = \mathrm{OA}\sin\mathrm{A} = \mathrm{FD} = \mathrm{OB}\sin\mathrm{B} = \mathrm{DE} = \mathrm{OC}\sin\mathrm{C};$$

$$\therefore \quad \mathrm{OA} : \mathrm{OB} : \mathrm{OC} = 1/\sin\mathrm{A} : 1/\sin\mathrm{B} : 1/\sin\mathrm{C} = 1/a : 1/b : 1/c.$$

Hence the point O may be found as follows :—

Let the bisectors of the angles at A meet BC in D and D′; on DD′ as diameter describe a circle. The two points of intersection of this circle, and the circles obtained by taking the other two sides instead of the side BC, are the points required.

§ 9. *Analytical Investigation.*

In fig. 24, let $\mathrm{OD} = \alpha$, $\mathrm{OE} = \beta$, $\mathrm{OF} = \gamma$, $\mathrm{DE} = \mathrm{EF} = \mathrm{FD} = l$; then

$$\mathrm{OE}^2 + \mathrm{OF}^2 + 2\mathrm{OE}.\mathrm{OF}\cos\mathrm{A} = l^2;$$

or, $$\beta^2 + \gamma^2 + 2\beta\gamma\cos\mathrm{A} = l^2; \tag{1}$$

similarly, $$\gamma^2 + \alpha^2 + 2\gamma\alpha\cos\mathrm{B} = l^2; \tag{2}$$

$$\alpha^2 + \beta^2 + 2\alpha\beta\cos\mathrm{C} = l^2. \tag{3}$$

On subtracting the second of these equations from the first we get the equation to one of the circles of § 7, namely,

$$\beta^2 - \alpha^2 + 2\gamma(\beta\cos\mathrm{A} - \alpha\cos\mathrm{B}) = 0; \tag{4}$$

and we get the other two by taking the other two differences.

Equation (4) may be expressed in the form

$$(b^2 - c^2)(a\beta\gamma + b\gamma\alpha + c\alpha\beta) + a(a\alpha + b\beta + c\gamma)(c\beta - b\gamma) = 0. \tag{5}$$

The radical axis of the three circles given by the equations (1), (2), (3), is

$$bc(b^2 - c^2)\alpha + ca(c^2 - a^2)\beta + ab(a^2 - b^2)\gamma = 0;$$

or, $$\alpha \sin(\mathrm{B} - \mathrm{C}) + \beta\sin(\mathrm{C} - \mathrm{A}) + \gamma\sin(\mathrm{A} - \mathrm{B}) = 0.$$

A simple geometrical construction may be given for this line,

namely (fig. 26) make $\angle$ ABQ = $\angle$ ACQ = A; let AQ meet BC in P; then P is the point where the radical axis meets BC.

The equation to the radical axis of the circle of equation (5) and the circumcircle of the triangle is obviously $c\beta - b\gamma = 0$. It may easily be shown by means of this equation that if the radical axis meets the side AB in O, then AO : OB = b^2 : a^2. This may also be proved without the use of the above equations, by calculating the segments AO and OB. It follows from this result that AO is one of the symmedians of the triangle.

§ 10. Generalization.

To find a point O (fig. 27) such that, if OD′, OE′, OF″, be the perpendiculars from O on the sides of ABC, the triangle D′E′F″ shall be similar to any given triangle DEF.

On AB, BC, CA, describe the triangles ABC′, A′BC, AB′C, similar to the triangle DEF (the triangles are named as they correspond to DEF); and let all these triangles be described externally or all internally on the sides of ABC. Then in both cases the circles circumscribing these triangles will be concurrent (in a point P); and the point inverse to this point of concurrence will be the point O required. The proof is almost identical with that given before for the special case when DEF is equilateral.

The point P may also be obtained as the point of concurrence of the lines AA′, BB′, CC′.

It should be noticed that, according to the construction given above, the point corresponding to D will lie in BC, the point corresponding to E in CA, and the point corresponding to F in AB. By making the triangles ABC′, AB′C, A′BC, similar to EFD, the vertices corresponding in this order, we may make the vertex corresponding to E lie in BC, that corresponding to F lie in CA, and that corresponding to E in AB; and by making other variations in the correspondence of the similar triangles, we may make the vertices corresponding to those of DEF lie in whichever sides of ABC we choose.

An analytical investigation, similar to that of § 9, may also be given.

Pappus on the Progressions.

By J. S. Mackay, LL.D.

[The present paper is a translation of the second part of the third book of Pappus's *Mathematical Collection*. Pappus's date is uncertain, but 300 A.D. may be taken as an approximation to it.

Throughout the translation I have used the word "progression" as a rendering of the Greek *μεσότης*, which has no English equivalent. The only other alternative was to employ the term *mediety*, from the Latin *medietas*.

The account of the various progressions given by Nicomachus, in his *Arithmetical Introduction*, differs somewhat from that of Pappus. I hope to have something to say about Nicomachus in a future paper.]

The second problem was this:

Figure 28.

Some other person said that the three progressions could be obtained in a semicircle thus. He described a semicircle ABC whose centre was E; taking any point D in AC, and drawing DB at right angles to EC, he joined EB, and from D drew DF perpendicular to it. The three progressions, he maintained, were exhibited in a simple manner in the semicircle; for EC was the arithmetical mean, DB the geometrical, and BF the harmonical.

That BD is the mean of AD, DC in a geometrical proportion, and EC the mean of AD, DC in an arithmetical progression is evident. For $\quad AD : DB = DB : DC;$

and $$AD : AD = AD - AE : EC - CD,$$
$$= AD - EC : EC - CD.$$

But how BF is the mean of the harmonical progression, or of what straight lines it is the mean, he has not said; but only that it is the third proportional of EB, BD, not knowing that the harmonical progression is formed from EB, BD, BF when they are in a geometrical proportion*. For it will be shown by us later on that $2EB + 3DB + BF$ is the greater extreme, $2DB + BF$ is the mean, and $DB + BF$ is the least extreme of the harmonical progression.

But first, in order that we may more fully discuss the proposed demonstration, one must treat of the three progressions, and, after

* This censure from Pappus seems to be quite undeserved.

this, of the progressions in a semicircle, then of the three others which, according to the ancients, are opposed to these, and lastly, in accordance with the views of more recent geometers, of the four devised by them; and how it is possible to find by means of a geometrical proportion each of the ten progressions.

The three progressions.

A progression differs from a proportion in this, that every proportion is a progression, but not conversely. For there are three progressions—the arithmetical, the geometrical, the harmonical.

A progression is called arithmetical when there are three terms and the mean exceeds one of the terms by the same as it is exceeded by the remaining term, as 6 with reference to 9 and 3; or when the first term is to itself as the first difference to the second.

A progression is called geometrical, that is a proportion strictly, when the mean is to one of the terms as the remaining term is to the mean, as 6 with reference to 12 and 3; and otherwise, when the first term is to the second as the first difference to the second.

A progression is harmonical when the mean exceeds one of the extremes by the same fraction as it is exceeded by the remaining extreme, as 3 with reference to 2 and 6; or when the first term is to the third as the first difference is to the second difference.

These things having been laid down, we shall find the three progressions together in the five minimum straight lines, after premising the following.

First, having given the straight lines AB, BC, let it be proposed to find the mean according to a geometrical proportion.

Figure 29.

Let CD be drawn at right angles, and let AB be bisected at E. With centre E let a circle be described through B cutting CD at D; join BD and cut off BF equal to it. BF is the required mean.

If DA be joined, it contains a right angle with BD, because both BE and EA are equal to DE. Now the angle at C is right; therefore the triangle ABD is equiangular to the triangle BCD, and accordingly the sides about their common angle at B are proportional. Hence AB : DB = BD : BC, and BD or BF is the mean between AB and BC.

Given AB and BF let it be proposed to find the less extreme.

Figure 30.

Let AB be bisected at E; with centre E let a circle be described through B, and let this circle be cut at D by a circle described through F with centre B. Let a perpendicular DC be drawn; then BC is the third proportional to AB, BF.

The proof is similar to that regarding the mean.

Given FB, BC let it be proposed to find the greater extreme.

Figure 31.

Let CH be drawn at right angles, and with centre B let a circle described through F cut CH at H. Join BH, and draw AH at right angles to it. Then AB is the third proportional to CB, BF.

This is obvious from what has been proved before.

Figure 32.

Again, let there be two straight lines AB, BC, and let DAE be at right angles to AB, so that AD is equal to AE. Let BD, ECF be joined, and from F let FG be drawn perpendicular to CB. Then $AB : BG = AB - BC : CB - BG$.

For $AB : BG = DA : FG$

$= AE : FG$, since $AE = AD$;

$= AC : CG$, on account of the triangles ACE, CFG being equiangular.

Now $AC = AB - BC$, and $CG = CB - BG$;

therefore $AB : BG = AB - BC : CB - BG$.

Figure 32.

But if the extremes AB, BG be given and we seek the mean, join BD, and from G draw FG at right angles. From F to E draw FCE, and we shall have CB the mean between AB and BG.

The proof is obvious.

Figure 33.

Given EB, BC we shall find the greater extreme by drawing from E, DEF at right angles, making $DE = EF$, joining BF, DC, and producing them to G.

For GH the perpendicular drawn from G to BC produced will cut off HB equal to what is sought.

Again, given two straight lines AB, C, of which AB is the greater, we shall find the equidifferent mean thus.

Figure 34.

Make $DB = C$, bisect DA at E, and make $F = EB$. It is obvious that F is the straight line sought.

Similarly if F, C be given, by adding their difference to F we shall have a straight line equal to AB.

Again, if AB, F be given, their difference subtracted from F will give C the third.

Figure 35.

If therefore F be the equidifferent mean of AB, C, the straight lines AB, F, C will form an arithmetical progression. As F is to C so make C to G; then the straight lines F, C, G will form a geometrical progression, that is, a proportion strictly. And if, according to what has been proved before, having two straight lines C, G, the greater of which is C, we make H such that

$$C : H = C - G : G - H,$$

then the straight lines C, G, H will form a harmonical progression. Now $AB : C = C : H$, AB and C being the extreme terms in the arithmetical progression, and C and H in the harmonical; there will therefore be five minimum straight lines containing the three progressions [and these may be incommensurable with one another].

Now let it be proposed to form the three progressions with the minimum five numbers, and according to what are called multiple, superparticular, and other ratios, unity being supposed indivisible. When the ratio of AB to C is 2, for instance, the minimum numbers which effect what is proposed will be 12, 9, 6, 4, 3; when the ratio is 3, the minimum numbers will be 18, 12, 6, 3, 2. And it is evident how with other ratios also, the minimum numbers for the three progressions must be found. Now if one should wish to express separately each of the progressions, that is clear from what has been previously written; the three terms of the arithmetical progression being in the minimum numbers 3, 2, 1, of the geometrical 4, 2, 1, and the numbers which, according to the given ratio, are fundamental being transformed into equimultiples and superparticulars and the rest. For example, if AB has to C the ratio of 2 to 1, instead of 2 we shall put 4, and instead of 1 we shall put 2. And since the mean between these must exceed and be exceeded by the same amount, the straight line F consists of 3 units. Now the ratio of F to C is that of 3 to 2, and if the ratio of C to G be made equal to it,

the problem is not done, because unity remains indivisible. Let everything then be tripled, and 12 is obtained for 4, 9 for 3, 6 for 2. The straight line G then becomes one of 4 units, and H manifestly of 3, and the numbers for the three progressions are 12, 9, 6, 4, 3.

So much, then, concerning the three progressions according to the ancients. Thence it is evident that it is possible to exhibit the three progressions together in a semicircle in the minimum six straight lines.

Figure 36.

Let a semicircle be described having BD perpendicular, and EB a radius, and DF perpendicular to EB. Through B draw HG touching the circle, produce EC to G, make BH equal to BG, and join DKH. Then in the harmonical progression EK is the mean between BE and EF, the greatest term being BE and the least EF.

Since the angles at B and F are right, DF is parallel to HG, and the triangle EBG is equiangular to the triangle EFD, and the triangle BHK to the triangle FKD;

therefore $\quad BE : EF = GB : FD,$

$\qquad\qquad = HB : FD,$ because $BG = BH$;

$\qquad\qquad = BK : KF.$

Now $\quad BK = BE - EK, \quad KF = KE - EF;$

therefore $\quad BE : EF = BE - EK : KE - EF.$

The straight lines BE, EK, EF then contain the harmonical progression, the mean being EK, the greatest BE, and the least EF. And AD, EC, CD were shown to contain the arithmetical progression, AD, BD, DC the geometrical. The three progressions therefore have been arranged in a semicircle.

Since Nicomachus,* the Pythagorean, and some others have spoken not only of the first three progressions, which are the most useful in the study of ancient authors, but also of the other three which were in vogue among the ancients, and since, in addition to these six, other four have been invented by more recent writers, we shall endeavour to speak of these somewhat carefully (?), following, however, the older writers who began from the greater term. . . . . . [The Greek text is here corrupt.]

For when the third term is to the first as the excess of the first term is to that of the second, they call the progression contra-harmonical.

---

* In his *Arithmetical Introduction*. Nicomachus's date is about 100 A.D.

But when the third term is to the second as the excess of the first is to that of the second, the progression is called the fifth and contro-geometrical, for some name it so.

When the second term is to the first as the excess of the first is to that of the second, the progression is called the sixth. It also is called contra-geometrical, because in the sequence of the ratios the order is reversed. Thus, according to them, there are six progressions.

By more recent writers, as we said, other four have been found, in some respect useful, and their discoverers employ their own definitions. For they call the excess of the first term above the second the first difference, that of the second above the third the second difference, and that of the first above the third the third difference, the greatest term being understood and spoken of, as we explained at the outset, as the first, the mean as the second, and the least as the third.

When the third difference is to the first as the second term to the third, they call the progression the seventh.

But, while the ratio of the differences remains the same, if it be as the first term to the second, they call the progression the eighth.

If the third difference be to the first as the first term to the third, they call the progression the ninth.

If the third difference be to the second as the second term to the third, they call the progression the tenth.

Having laid down these definitions, we shall explain the origins of the ten progressions, as we said, by means of a geometrical proportion.

The geometrical progression then, since it takes its first origin from equality, will constitute both itself and the other progressions, showing, as saith the divine Plato, that the nature of proportion is the cause of the harmony of all things, and of a rational and ordered creation. For he says that there is one bond of all the sciences. Now the cause of creation and the bond by which all created things are held together is the divine nature of proportion. The constitution of the ten progressions will be shown by means of the geometrical proportion, the following being premised.

Let there be three terms A, B, C proportional, and let $D = A + 2B + C$, $E = B + C$, $F = C$, then the terms D, E, F are proportional.

Since $$A : B = B : C,$$

by composition $\quad A+B:B=B+C:C$;
therefore the sum of the antecedents is to the sum of the consequents in the same ratio,
that is, $\quad A+2B+C:B+C=B+C:C$.
Now $\quad D=A+2B+C,\ E=B+C$, and $F=C$;
therefore $\quad D:E=E:F$.

Hence if A, B, C be supposed equal, D, E, F will be in a double proportion; for $A+2B+C=2(B+C)$ and $B+C=2C$. But if A, B, C be supposed to be in a double proportion and A to be the greatest term among them, D, E, F will be in a triple proportion; and if A be the least term, D, E, F will be in sesquialterate proportion. For if $A=2B$, then $A+B=3B$; and if $A=\frac{1}{2}B$, then $A+B=\frac{3}{2}B$.
And so from the succeeding ratios, the numbers which follow, both multiples and superparticulars, will be found.

And again, if A, B, C were units, the geometrical progression formed by D, E, F would be said to be in minimum numbers 4, 2, 1.

[The translators of Pappus, Commandinus, and Hultsch, in the belief that a lacuna exists here in the Greek text, have inserted a proposition showing how the arithmetical progression is constituted by means of a proportion.

Commandinus puts

$$D=2A+2B+C,\ E=A+B+C,\ F=C,$$

and finds the minimum numbers 5, 3, 1. This, however, does not agree with the entries for the arithmetical progression in the table at the end, as given in his edition of Pappus.

Hultsch puts

$$D=2A+3B+C,\ E=A+2B+C,\ F=B+C,$$

and finds the minimum numbers 6, 4, 2. These are the numbers given for the arithmetical progression in the tables of all the MSS. of Pappus which I have examined.]

The harmonical progression is thus constituted by means of a proportion.

Let three terms A, B, C be supposed proportional,
and let $\quad D=2A+3B+C,\ E=2B+C,\ F=B+C$;
then D, E, F form the harmonical progression.

Since A, B, C are proportional,
therefore $\quad 2A+B:B=2B+C:C$.
Taking the sum of the antecedents and the sum of the consequents,

$$2A + 3B + C : B + C = 2A + B : B,$$

that is $$D : F = 2A + B : B.$$

Now $$D - E = 2A + B, \text{ and } E - F = B;$$

and when $$D : F = D - E : E - F,$$

the progression is harmonical.

And it is evident, if A, B, C be supposed to be units, that the harmonical progression is constituted in minimum numbers 6, 3, 2.

The contra-harmonical progression is thus constituted from a proportion.

Let the terms A, B, C be supposed proportional,

and let $$D = 2A + 3B + C, \; E = 2A + 2B + C, \; F = B + C;$$

then D, E, F form the said progression.

For again, similarly to what has been shown before,

$$D : F = 2A + B : B.$$

And $$E - F = 2A + B, \; D - E = B;$$

therefore $$F : D = D - E : E - F,$$

which is what characterises the contra-harmonical progression.

And it is evident, if A, B, C be supposed units, that the progres sion is constituted in minimum numbers 6, 5, 2.

The fifth progression is thus constituted from a proportion.

Let the three terms A, B, C be supposed proportional,

and let $$D = A + 3B + C, \; E = A + 2B + C, \; F = B + C;$$

then D, E, F are in the fifth progression.

Since, on account of the proportion,

$$A + B : B = B + C : C$$

taking the sum of the antecedents and the sum of the consequents,

$$A + 2B + C : B + C = A + B : B,$$

that is $$E : F = A + B : B.$$

Now, $$E - F = A + B, \text{ and } D - E = B;$$

therefore $$F : E = D - E : E - F,$$

which is what happens in the fifth progression.

And if A, B, C be supposed units, the progression would be said to be in minimum numbers 5, 4, 2.

The sixth progression is thus constituted from a proportion.

Let the proportion of the terms A, B, C be the same,

and let $$D = A + 3B + 2C, \; E = A + 2B + C, \; F = A + B - C;$$

then D, E, F form the proposed progression.

Since, on account of the proportion,

$$A+2B : A+B = B+2C : B+C;$$

taking the sum of the antecedents and the sum of the consequents,

$$A+3B+2C : A+2B+C = B+2C : B+C,$$

that is $$D : E = B+2C : B+C.$$

Now $$E-F = B+2C, \text{ and } D-E = B+C;$$

therefore $$E : D = D-E : E-F,$$

so that D, E, F form the sixth progression.

And if A, B, C be supposed units, it is similarly constituted in minimum numbers 6, 4, 1.

[Here, also, a lacuna has been presumed to exist in the Greek text by Pappus's commentators.

Commandinus puts

$$D = A+2B+2C,\ E = A+B+C,\ F = B+C,$$

and finds the minimum numbers 5, 3, 2. This again does not agree with the entries for the seventh progression in the table, as given in his edition.

Hultsch puts

$$D = A+B+C,\ E = A+B,\ F = C,$$

and finds the minimum numbers 3, 2, 1.

I have given the table at the end, which is much corrupted in the MSS., as it exists in Hultsch's edition, vol. I., pp. 102-103, although I have long entertained some suspicion of its genuineness, as well as of the need for filling up the presumed lacunae.]

The eighth progression is thus constituted from a proportion.

Let the three terms A, B, C be supposed proportional,

and let $$D = 2A+3B+C,\ E = A+2B+C,\ F = 2B+C;$$

then D, E, F are according to the eighth progression.

Since, on account of the proportion,

$$2A+B : A+B = 2B+C : B+C;$$

taking the sum of the antecedents and the sum of the consequents,

$$2A+3B+C : A+2B+C = 2A+B : A+B,$$

that is $$D : E = 2A+B : A+B.$$

Now $$D-F = 2A+B, \text{ and } D-E = A+B;$$

therefore $$D : E = D-F : D-E,$$

which constitutes the eighth progression.

And if A, B, C be supposed units, it would be said to be in minimum numbers 6, 4, 3.

The ninth progression is thus constituted from a proportion.

Let A, B, C be supposed proportional,

and let $D = A + 2B + C$, $E = A + B + C$, $F = B + C$;
then D, E, F contain the ninth progression.

Since $$A + B : B = B + C : C;$$
taking the sum of the antecedents and the sum of the consequents,
$$A + 2B + C : B + C = A + B : B,$$
that is $$D : F = A + B : B.$$
Now $$D - F = A + B, \text{ and } D - E = B;$$
therefore $$D : F = D - F : D - E,$$
which is the characteristic of the ninth progression.

And if A, B, C be similarly supposed units, the minimum numbers 4, 3, 2 contain the progression.

The tenth progression is thus constituted from a proportion.

Again let the three A, B, C be proportional,
and let $D = A + B + C$, $E = B + C$, $F = C$;
then D, E, F are according to the tenth progression.

For $$B + C : C = A + B : B,$$
that is $$E : F = A + B : B.$$
Now $$D - F = A + B, \text{ and } E - F = B;$$
therefore $$E : F = D - F : E - F,$$
which happens in the tenth progression.

And if A, B, C be supposed units, the minimum numbers 3, 2, 1 form the progression.

For the sake of convenience there are set out the successive numbers by which each term of the proportion is multiplied so as to form each progression, and beside them are placed the minimum numbers containing the progressions. For instance, in the table of the sixth progression the first row 1, 3, 2 means this, that the first term of the proportion taken once, the second thrice, and the third twice, complete the first term of the progression; the second row of the table 1, 2, 1 means that the first term of the proportion taken once, the second twice, and the third once, complete the second term of the progression. The third row of the table in the remaining progressions is composed simply as has been described; exceptionally, however, in this progression the row 1, 1, 1 signifies, as has been said before, that the third term of the progression is obtained from the difference by which the first term of the proportion taken once, and the second taken once, exceed the third term taken once. In the third part of the table the numbers 6, 4, 1 contain the progression itself. Let similar things be understood regarding the remaining tables.

PROGRESSIONS.	COEFFICIENTS OF THE TERMS A, B, C.	THREE MINIMUM NUMBERS CONTAINING THE PROGRESSIONS.
Arithmetical	2 3 1 1 2 1 1 1	6 4 2
Geometrical	1 2 1 1 1 1	4 2 1
Harmonical	2 3 1 2 1 1 1	6 3 2
Contra-harmonical	2 3 1 2 2 1 1 1	6 5 2
Fifth	1 3 1 1 2 1 1 1	5 4 2
Sixth	1 3 2 1 2 1 1 1 1	6 4 1
Seventh	1 1 1 1 1 1	3 2 1
Eighth	2 3 1 1 2 1 2 1	6 4 3
Ninth	1 2 1 1 1 1 1 1	4 3 2
Tenth	1 1 1 1 1 1	3 2 1

Seventh Meeting, May 11th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

"Vortex Rings in a Compressible Fluid."

By C. CHREE, M.A.

In a paper recently printed in the Society's *Proceedings*, I considered the effect of compressibility in the fluid on the motion of straight vortices; the present paper treats of circular vortex rings in a compressible fluid. The circle passing through the centres of the circular cross sections of the vortex filament will be called the "circular axis," and the perpendicular to the plane of the circular axis through its centre, the "axis" of the vortex. In the notation employed, a denotes the radius of the circular axis, and e that of the cross section of the filament, while ω represents vorticity, and ρ density. It is also convenient to denote the area of the cross section—*i.e.*, πe^2, by σ. Following Helmholtz, it will be supposed that e/a is always very small, and that the cross section is truly circular. Certain small inconsistencies in the ordinary theory following from this last assumption will be pointed out, though they do not seem seriously to affect the general applicability of the results. The axis of the vortex ring is taken as axis of z, and z, r, θ are the ordinary cylindrical co-ordinates. It is also convenient to denote by r' the distance of a point from the circular axis of a ring, and by ψ the inclination of this distance to the plane of the circular axis. The effects of the vorticity and variation in density may be considered separately.

The components of vorticity at any point of the ring are $\xi = -\omega\sin\theta$, $y = \omega\cos\theta$. There is obviously symmetry about oz, and the velocity at any point can be resolved into w parallel to oz and u along the perpendicular on oz. Since e/a is very small, we shall, following the common practice in calculating the velocity, regard the

vortex filament as concentrated in the circular axis, and so in Lamb's formulæ* replace the vortex element $dxdydz$ by $\frac{ma}{\omega}d\theta$; where $m, = \pi e^2\omega$, is the strength of the vortex. Supposing the origin taken in the instantaneous position of the centre of the circular axis, we find from these formulæ for the velocity in the fluid at the point r, θ, z outside the vortex

$$w = \frac{ma}{2\pi}\int_0^{2\pi} \frac{(a - r\cos\theta)d\theta}{(z^2 + a^2 + r^2 - 2ar\cos\theta)^{\frac{3}{2}}} \quad \cdots \quad \cdots \quad (1),$$

$$u = \frac{ma}{2\pi}\int_0^{2\pi} \frac{z\cos\theta d\theta}{(z^2 + a^2 + r^2 - 2ar\cos\theta)^{\frac{3}{2}}} \quad \cdots \quad \cdots \quad (2).$$

We may at once transform (1) into

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\frac{1}{\sqrt{z^2 + (r+a)^2}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}\right] \quad \cdots \quad \cdots \quad (3),$$

$$\text{where } k^2 = \frac{4ra}{z^2 + (r+a)^2},\ 2\phi = \pi - \theta \quad \cdots \quad \cdots \quad (4).$$

Thus if, as usual, F_1 denote the complete elliptic integral of the first order,

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \cdots \quad \cdots \quad (5).$$

When k is nearly unity, an approximate value is $F_1(k) = \log(4/k_1)$,

$$\text{where } k_1{}^2 = 1 - k^2 = \frac{z^2 + (r-a)^2}{z^2 + (r+a)^2} \quad \cdots \quad \cdots \quad (6).$$

Thus, for points near the surface of the filament, where z and $r - a$ are both very small, an approximate value is

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\}\right].$$

This gives

$$w = \frac{2ma}{\pi}\{z^2 + (r+a)^2\}^{-\frac{3}{2}}\left[(r+a)\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\} - \frac{2r(z^2 + r^2 - a^2)}{z^2 + (r-a)^2}\right] \cdots \quad \cdots \quad (7).$$

For points outside, but in contact with the surface of the filament, we have $r = a + e\cos\psi, z = e\sin\psi$. Substituting these values in (7) and

* "Motion of Fluids"—Equations (15), p. 152.

retaining the principal terms, we get for the velocity in the fluid just outside the filament

$$w=\frac{m}{2\pi a}\left\{\log\frac{8a}{e}-1+\cos^2\psi-\frac{2a\cos\psi}{e}\right\} \quad \ldots \quad \ldots \quad (8).$$

The first and last are much the most important terms.

The most important term omitted is $-\frac{me\cos\psi}{2\pi a^2}\log\frac{8a}{e}$.

From (2), using k and ϕ in the same sense as before, we easily find

$$u=\frac{m}{\pi r}\frac{d}{dz}\left[\{z^2+(r+a)^2\}^{\frac{1}{2}}\mathrm{E}_1(k)-\frac{z^2+r^2+a^2}{\{z^2+(r+a)^2\}^{\frac{1}{2}}}\mathrm{F}_1(k)\right] \quad \ldots \quad (9);$$

where E_1 is the complete elliptic integral of the second order.

For points near the surface of the filament, we saw that approximately

$$\mathrm{F}_1(k)=\log(4/k_1);$$

also

$$\mathrm{E}_1(k)=(1-k^2)\left\{\mathrm{F}_1(k)+\frac{kd}{dk}\mathrm{F}_1(k)\right\} \quad \ldots \quad \ldots \quad (10).*$$

From its approximate value we get $k\frac{d\mathrm{F}_1(k)}{dk}=\frac{k^2}{k_1^2}=\frac{4ar}{z^2+(r-a)^2}$;

thus an approximate value is

$$\mathrm{E}_1(k)=\frac{z^2+(r-a)^2}{z^2+(r+a)^2}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\}+\frac{4ar}{z^2+(r-a)^2}\right] \quad \ldots \quad (11).$$

Substituting these values in (9), carrying out the differentiations and reducing, we obtain the approximate value

$$u=\frac{2maz}{\pi}\left\{z^2+(r+a)^2\right\}^{-\frac{3}{2}}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\}\right.$$
$$\left.-2+\frac{4ar}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (12)$$

Using the same notation as before, and retaining the principal terms, we find for the velocity in the fluid just outside the filament

$$u=\frac{m}{\pi e}\sin\psi-\frac{m}{2\pi a}\sin\psi\cos\psi \quad \ldots \quad \ldots \quad (13).$$

The most important term omitted is $\frac{me\sin\psi}{4\pi a^2}\log\frac{8a}{e}$.

From (8) and (13) we see that the velocity in the fluid just outside the vortex is composed of the two components

$$w=\frac{m}{2\pi a}\left\{\log\frac{8a}{e}-1\right\} \quad \ldots \quad \ldots \quad (14),$$

---

* Cayley's "Elliptic Functions."

parallel to the axis oz, and

$$\tau = \frac{m}{\pi e}\left(1 - \frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \qquad (15),$$

tangential to the surface of the filament in the plane containing oz.

The former component (14) is a motion *en masse*, the same at every point of the surface of the filament, and is shared by the vortex and the fluid bordering on it; the latter (15) represents the velocity of circulation of the fluid round the circular axis. This velocity is slightly greater on the concave, or inner, side of the filament, and less on the convex side than in the case of a straight vortex of the same strength and cross section.

This is one of the small inconsistencies already referred to; for if the vorticity ω be constant throughout the filament and the section truly circular, the velocity of circulation in the fluid just inside the surface of the filament must be ωe, while from (15) the fluid just outside has its velocity of circulation $= \omega e\left(1 - \frac{e}{2a}\cos\psi\right)$. There is thus a very slight absence of continuity in the motion on crossing the surface. In a perfectly frictionless fluid this may seem of absolutely no importance, but the following reasoning shows that an inconsistency of a precisely similar character exists in the hypothesis that ω can be constant throughout a truly circular section.

Using r' in the sense already indicated, let us not assume ω to be constant, but still suppose the velocity perpendicular to r'. Consider the elementary ring formed by the revolution about oz of the element $r'd\psi dr'$ of the cross section of the filament. The volume of this ring is $2\pi(a + r'\cos\psi)r'd\psi dr'$, and the areas of the two surfaces through which alone flow takes place are each $2\pi(a + r'\cos\psi)dr'$. The velocity is normal to these surfaces and equal to $\omega r'$. Thus, by the equation of continuity, if the fluid be incompressible, we get

$$\frac{d}{d\psi}\left[\left(1 + \frac{r'}{a}\cos\psi\right)\omega r'\right] + \frac{d}{dt}\left[\left(1 + \frac{r'}{a}\cos\psi\right)r'\right] = o.$$

Since $\frac{d\psi}{dt} = \omega$, this gives at once

$$\omega\left(1 + \frac{r'}{a}\cos\psi\right)^2 = \text{constant}.$$

Neglecting terms in $(r'/a)^2$, and denoting the mean value of the vorticity by Ω, this gives

$$\omega = \Omega\left(1 - \frac{2r'}{a}\cos\psi\right).$$

Thus, our fundamental hypothesis that ω and e are constant when logically carried out requires us to neglect the second term in (15), *i.e.*, velocities of order m/a. This consequently would lead us to neglect the second term in (14) also. This may explain the slight divergence in the results obtained by Prof. J. J. Thomson,* Mr T. C. Lewis,† and Mr W. M. Hicks.‡ The two former obtain the result (14), but the latter differs in the value of the small term.

We have next to consider the velocity due to variation in the density ρ of the fluid. From the equations (6) and (10) of Chap. VI. of Lamb's Treatise, it follows that the velocity due to change of density is expressed by the same formula as the force due to a gravitating mass of density $-\frac{1}{4\pi\rho}\frac{\delta\rho}{\delta t}$. If the ring be of small cross section σ we may in calculating the velocity, to the same degree of accuracy as when treating the vorticity, regard it as equal to the force due to a mass of line density $-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}$ concentrated in the circular axis $r = a$ of the ring. If ρ varied with the distance from the circular axis, but was independent of ψ, the accuracy would not be seriously affected.

If w_1 and u_1 denote the component velocities parallel and perpendicular to the axis of the ring, due to the variation in density alone, then from the above remarks it follows that

$$w_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}az\int_o^\pi \frac{d\theta}{(z^2 + r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} \quad \dots \quad \dots \qquad (16),$$

$$u_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}a\int_o^\pi \frac{(r - a\cos\theta)d\theta}{(z^2 + r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} \quad \dots \quad \dots \qquad (17).$$

Thence, k having its previous meaning, we find

$$w_1 = \frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\frac{d}{dz}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \dots \quad \dots \qquad (18).$$

To the same degree of approximation as in the case of vorticity, we

---

* "Motion of Vortex Rings"—Equation (41), p. 33.

† "Quarterly Journal of Mathematics," XVI., 1879, pp. 338-347.

‡ "Philosophical Transactions," 1884, Part I., and 1885, Part II.

have for the velocity in the fluid comparatively near the surface of the filament

$$w_1 = -\frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}az\{z^2+(r+a)^2\}^{-\frac{3}{2}}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\} + \frac{4ra}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (19).$$

For points just outside the ring, using the previous notation and retaining the principal terms, we find

$$w_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\sin\psi\left(1-\frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \quad (20).$$

The most important term neglected is $-\dfrac{\sigma}{8\pi\rho}\dfrac{\delta\rho}{\delta t}\dfrac{e}{a^2}\sin\psi\log\dfrac{8a}{e}$.

From (17) we find

$$u_1 = \frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\frac{d}{dr}\left[\{z^2+(r+a)^2\}^{-\frac{1}{2}}\mathrm{F}_1(k)\right] \quad \ldots \quad \ldots \quad (21);$$

and thence for points comparatively near the ring

$$u_1 = -\frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\{z^2+(r+a)^2\}^{-\frac{3}{2}}\left[(r+a)\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\} + 2a\frac{(r^2-a^2-z^2)}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (22).$$

For points just outside the ring we thence obtain the approximate value

$$u_1 = -\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\left[\frac{1}{a}\left(\log\frac{8a}{e}-1\right)+\frac{2}{e}\cos\psi\left(1-\frac{e}{2a}\cos\psi\right)\right] \quad \ldots \quad \ldots \quad (23).$$

The most important term neglected is $\dfrac{\sigma}{4\pi\rho}\dfrac{\delta\rho}{\delta t}\dfrac{e}{a^2}\cos\psi\log\dfrac{8a}{e}$.

From (20) and (23) we see that the velocity in the fluid just outside the filament is composed of the two components

$$u_1 = -\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\left\{\log\frac{8a}{e}-1\right\} \quad \ldots \quad \ldots \quad (24),$$

perpendicular to the axis oz, *i.e.*, tending to increase the radius a of the circular axis, and

$$v_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\left(1-\frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \quad (25),$$

normal to the surface of the filament, *i.e.*, tending to increase e.

The first component u_1 represents a motion of the ring and surrounding fluid *en masse;* the second v_1 gives the rate of increase in the radius of the cross section consequent on the change in density. The slight variation in the rate of increase of e in different directions

is a phenomenon exactly similar to that illustrated by the existence of the small term in (15). A consideration of the equation of continuity shows us, precisely as in the parallel case in vorticity, that if the density be uniform over the cross section it cannot vary so that the cross section remain truly circular, unless velocities of the order $\frac{\sigma}{\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}$ be negligible. In this case the second terms in both (24) and (25) must be neglected. It would even seem at first sight that the equation of continuity was inconsistent with the existence of the principal term of (24). For, since the mass of the ring is constant, $e^2 a\rho$ must be constant, and so

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t}+\frac{2}{e}\frac{\delta e}{\delta t}=-\frac{1}{a}\frac{\delta a}{\delta t} \quad \dots \quad \dots \qquad (26).$$

But $v_1=\frac{\delta e}{\delta t}$, and so, retaining only the principal term of (25) and putting $\sigma=\pi e^2$, we get $\frac{1}{\rho}\frac{\delta\rho}{\delta t}+\frac{2}{e}\frac{\delta e}{\delta t}=0 \quad \dots \quad \dots \qquad (27).$

Thus it might be thought from (26) that $\frac{\delta a}{\delta t}$ must vanish.

The true explanation is that $\frac{\delta a}{\delta t}$ does not vanish, but $\frac{1}{a}\frac{\delta a}{\delta t}$ is of an order of small quantities we agreed to neglect when we came to the conclusion that the second terms in (24) and (25) were negligible. To this degree of approximation, then, we see from (27) that $\sigma\rho$ is constant, and we may replace $-\frac{\sigma}{\rho}\frac{\delta\rho}{\delta t}$ by $\frac{\delta\sigma}{\delta t}$.

Combining the effects of vorticity and change of density, and retaining only the terms consistent with an exactly circular cross section, we finally obtain for the velocities of a thin filament

$$\left.\begin{aligned} w&=\frac{m}{2\pi a}\log\frac{8a}{e} \\ u&=\frac{1}{4\pi a}\frac{\delta\sigma}{\delta t}\log\frac{8a}{e} \end{aligned}\right\} \quad \dots \quad \dots \qquad (28).$$

The circular axis of the ring moves on the surface formed by the revolution about oz of the curve whose differential equation is

$$\frac{dz}{dx}=\frac{w}{u}=2m\Big/\frac{\delta\sigma}{\delta t} \quad \dots \quad \dots \qquad (29).$$

If the rate of increase of the cross section be uniform this forms part

of a right circular cone whose vertical angle is $2\tan^{-1}\left(\frac{1}{2m}\frac{\delta\sigma}{\delta t}\right)$.

It is easily seen that the case of a ring vortex in presence of an infinite plane, whether inclined or not to the plane of the circular axis, can be treated by the introduction of an "image" ring on the other side of the plane. The position and direction of rotation in the image were indicated in my previous paper; the cross section and density must be the same as at corresponding points in the real ring. However, the ring will remain circular with its circular axis in one plane only when it is parallel to the infinite plane, and the formulæ obtained above will be most usefully employed when the distance c of the ring from the plane is small compared to the radius a, though large compared to e. This case we proceed to treat.

Let us take the origin where the infinite plane is intersected by the common axis of the rings.

The components of the velocity at any point due to the real ring may be got from the preceding formulæ by writing $z-c$ for z, while the components of the velocity due to the image ring require the substitution of $z+c$ for z, and $-m$ for m.

For the velocity in the fluid immediately surrounding the ring the effect of the ring itself is given by (8), (13), (20), and (23), while the effect of the image may be got from (7), (12), (19), and (22) by writing $-m$ for m, $a+e\cos\psi$ for r and $2c+e\sin\psi$ for z. Combining the effects and retaining only the principal terms in accordance with the remarks already made as to the probable degree of accuracy of the method, I find

$$w=-\frac{m\cos\psi}{\pi e}-\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\sin\psi+\frac{m}{2\pi a}\log\frac{2c}{e}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{c}\quad\dots\quad\dots\qquad(30),$$

$$u=\frac{m\sin\psi}{\pi e}-\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\cos\psi-\frac{m}{2\pi c}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\log\left(\frac{32a^2}{ce}\right)\quad\dots\qquad(31).$$

The two first terms in both (30) and (31) represent the velocity of circulation and the rate of increase of the radius of the cross section, while the two last terms in each represent a motion *en masse* shared by the ring and the surrounding fluid. Thus the velocity of the ring in the direction of the normal drawn *from* the infinite plane is

$$w=\frac{m}{2\pi a}\log\frac{2c}{e}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{c}\quad\dots\quad\dots\qquad(32),$$

while the rate of increase in the radius of the ring is

$$u = \frac{\delta a}{\delta t} = -\frac{m}{2\pi c} - \frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\log\left(\frac{32a^2}{ce}\right) \quad \dots \quad \dots \qquad (33).$$

Neglecting at first any change in ρ we have from (32) and (33)

$$\frac{\delta c}{\delta t} = \frac{m}{2\pi a}\log\frac{2c}{e} \quad \dots \quad \dots \qquad (34),$$

$$\frac{\delta a}{\delta t} = -\frac{m}{2\pi c} \quad \dots \quad \dots \qquad (35);$$

while $ae^2 = \text{constant}$.

It follows that $\dfrac{\delta^2 c}{\delta t^2} = \dfrac{m^2}{4\pi^2 a^2 c}\left\{\log\left(\dfrac{2c}{e}\right)^2 - \frac{1}{2}\right\}$.

Thus $\dfrac{\delta^2 c}{\delta t^2}$ is always positive, for c must be greater than e and so $\log\left(\dfrac{2c}{e}\right)^2$ greater than $\frac{1}{2}$. Hence if m be negative, or the vortex be approaching the plane, its rate of approach continually diminishes; while if m be positive, or the vortex be receding, its rate of retreat continually increases so long as (34) and (35) apply.

From (35) we see that the aperture, $2a$, of the vortex increases or diminishes continually according as it is approaching to or receding from the plane.

Considering next the effect of the variation in density alone, we see from (32) and (33) that if its density be increasing the ring approaches the plane with a continually diminishing aperture, while if the density be diminishing the ring recedes from the plane with a continually increasing aperture. The exact relation between the variations in a, e and ρ is given by (26), but to the degree of accuracy obtained here this may, as in the case of a solitary ring, be replaced by (27). Using (27) in (32), and replacing σ by πe^2, we find

$$\frac{\delta c}{\delta t} = \frac{e}{2c}\frac{\delta e}{\delta t} = \frac{1}{4\pi c}\frac{\delta\sigma}{\delta t};$$

whence it follows that $c^2 - \dfrac{1}{2}e^2 = c^2 - \dfrac{\sigma}{2\pi} = \text{constant} \quad \dots \quad \dots \qquad (36).$

In the cases to which our formulæ can be satisfactorily applied e/c is small, and so the total increase or diminution in the distance of the ring from the plane due to change in density alone is also small. Thus, in general, the effects of the vorticity will be much more important than the effects of variation in density. We conclude that if a vortex ring approach an infinite plane, its rate of approach is slightly greater if its density be increasing, and slightly less if its

density be diminishing, than it would be if the density remained constant.

The most important terms due to the action of the image ring which we have neglected in (30) and (31) respectively may without much difficulty be found to be

$$\left.\begin{aligned} w_2 &= \mathrm{A}\cos(\psi - a) \\ u_2 &= \mathrm{A}\sin(\psi - a) \end{aligned}\right\} \quad \cdots \quad \cdots \quad (37);$$

where, for shortness,

$$\left.\begin{aligned} \mathrm{A} &= \frac{e}{4\pi c^2}\left\{m^2 + \left(\frac{\sigma}{2\rho}\frac{\delta\rho}{\delta t}\right)^2\right\}^{\frac{1}{2}} \\ a &= \tan^{-1}\left(\frac{\sigma}{2m\rho}\frac{\delta\rho}{\delta t}\right) \end{aligned}\right\} \quad \cdots \quad \cdots \quad (38).$$

Since, as already explained, we are not warranted in retaining velocities of order $\frac{m}{a}$ it follows that the above terms represent an appreciable effect only when the vortex approaches so close to the plane that ae/c^2 becomes large.

From (37) it follows that the cross section of the filament tends to become slightly elliptical, the axes of the ellipse making with the infinite plane the angles $\frac{a}{2} \pm \frac{\pi}{4}$. These axes are thus equally inclined to the plane when the fluid is incompressible. When the deviation of the cross section from the circular form becomes appreciable the accuracy of the preceding formulæ will be lessened, and they can certainly not be applied to the case of a vortex whose distance from an infinite plane is of the same order of quantities as the diameter of its cross section.

If in (32) – (36) we write $c/2$ for c, we get the case of two precisely equal ring vortices, with vorticities, however, in opposite directions, at a distance c. If we suppose a to become infinite, we deduce formulæ applicable to the case of straight vortex filaments. In particular, it will be noticed that (36) leads at once to a special case of the formula obtained in my previous paper for the distance of two straight vortices.

Similitude and Inversion.

By J. S. MACKAY, LL.D.

The following paper contains little that can be regarded as new mathematical information. It aims only at showing, or rather at emphasising, the correspondence which exists between two geometrical theories which are related to each other in the same way as the arithmetical theories of multiplication and division. Such value, therefore, as it possesses is primarily pedagogical.

Attention should perhaps be drawn to the (unusual) use of the word "similar" in the sense of "similar and similarly situated," or "homothetic." The word homothesis (French geometers have adopted a curious form *homothétie*) is not naturalised in English; otherwise homothesis and homothetic might have been used instead of similitude and similar. The pair, similitude and homothetic, are somewhat incongruous.

The term antiparallel (said to have been first employed with a definite geometrical meaning by Antoine Arnauld in 1667) was not uncommon in English mathematical writings about a century ago. It is again, and deservedly, coming into use, and the following definition of it may be given:

Two straight lines intersecting the sides of an angle, or its vertically opposite angle, are antiparallel when the first straight line makes with one of the sides of the given angle an angle equal to that which the second straight line makes with the other side.

Figure 37′.

Thus, if $\angle OM'P' = \angle OPM$, then M′P′ is antiparallel to MP with respect to $\angle O$.

The following are some properties connected with antiparallels, the proof of which need not be given here.

(1) If M′P′ is antiparallel to MP with respect to $\angle O$, then MM′ is antiparallel to PP′.

(2) The four points M, M′, P, P′ are concyclic.

(3) The rectangles OM·OM′, OP·OP′ are equal.

(4) The triangles OMP, OM′P′ are similar.

SIMILITUDE.

§ 1. *Definition.*—If three collinear points O, P, P′ be given, any two of them may be considered similar to each other, and the third may be taken as their centre of similitude. The ratio of the distances of the third point from the other two is called the ratio of similitude.

Thus, if O be chosen as centre of similitude, and P′ be considered similar to P, the ratio of similitude is OP : OP′.

When P and P′ are on the same side of O, the ratio of similitude is positive, since OP and OP′ are drawn in the same direction; when P and P′ are on opposite sides of O, the ratio of similitude is negative, since OP and OP′ are drawn in opposite directions.

When a given point P is variable, that is, when it occupies a series of consecutive positions, the point P′ similar to it will also occupy a series of consecutive positions; in other words, when a given point P describes a certain curve, the similar point P′ will describe the similar curve.

§ 2. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the point similar to a given point P.

Figure 37.

Through O draw any straight line OM, and make OM $= r$, OM′ $= r'$. M and M′ will be on the same side of O or on opposite sides of O, according as $r : r'$ is positive or negative. Join MP, OP, and through M′ draw M′P′ parallel to MP, and meeting OP in P′.

Then P′ is similar to P.

For OP : OP′ = OM : OM′ $= r : r'$.

§ 3. Given two points P, P′ similar to each other, and a ratio of similitude $r : r'$, to find the centre of similitude.

The centre of similitude is determined by joining PP′ and dividing it externally or internally at O so that the segments OP, OP′ may have the given ratio.

§ 4. Given two pairs of similar points P and P′, Q and Q′, to find the centre of similitude.

Case 1. When the four points are not collinear.

INVERSION.

§ 1′. *Definition.*—If three collinear points O, P, P′ be given, any two of them may be considered inverse to each other, and the third may be taken as their centre of inversion. The rectangle under the distances of the third point from the other two is called the rectangle of inversion.

Thus, if O be chosen as centre of inversion, and P′ be considered inverse to P, the rectangle of inversion is $OP \cdot OP'$.

When P and P′ are on the same side of O, the rectangle of inversion is positive, since OP and OP′ are drawn in the same direction; when P and P′ are on opposite sides of O, the rectangle of inversion is negative, since OP and OP′ are drawn in opposite directions.

When a given point P is variable, that is, when it occupies a series of consecutive positions, the point P′ inverse to it will also occupy a series of consecutive positions; in other words, when a given point P describes a certain curve, the inverse point P′ will describe the inverse curve.

§ 2′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$ to find the point inverse to a given point P.

Figure 37′.

Through O draw any straight line OM, and make $OM = r$, $OM' = r'$. M and M′ will be on the same side of O or on opposite sides of O, according as $r \cdot r'$ is positive or negative. Join MP, OP, and through M′ draw M′P′ antiparallel to MP with respect to angle MOP, and meeting OP in P′.

Then P′ is inverse to P.

For $OP \cdot OP' = OM \cdot OM' = r . r'$.

§ 3′. Given two points P, P′ inverse to each other, and a rectangle of inversion $r \cdot r'$, to find the centre of inversion.

The centre of inversion is determined by joining PP′ and dividing it externally or internally at O so that the segments OP, OP′ may contain the given rectangle.

§ 4′. Given two pairs of inverse points P and P′, Q and Q′, to find the centre of inversion.

Case 1. When the four points are not collinear.

Join PP', QQ', and let them intersect at O. Then O is the centre of similitude.

This follows from § 1.

Case 2. When the four points are collinear.

FIRST METHOD.

Figure 38.

Take any point M not collinear with the four points, and join PM, QM.
Through P′ draw a straight line parallel to PM; through Q draw a straight line parallel to QM; and let these straight lines intersect at M′.
MM′ will intersect PP′ at O, the centre of similitude.

For
$$OP : OP' = OM : OM',$$
$$= OQ : OQ'.$$

SECOND METHOD.

Figure 39.

Take any point M not collinear with the four points, and join PM, QM.
Through Q′ describe a circle passing through P, M; through P′ describe a circle passing through Q, M; and let these circles intersect at M′.
MM′ will intersect PP′ at O, the centre of similitude.

For
$$OP \cdot OQ' = OM \cdot OM'.$$
$$= OQ \cdot OP';$$
therefore
$$OP : OP' = OQ : OQ'.$$

§ 5. It will be seen from the two preceding methods of solution that, when two pairs of similar points happen to be collinear, two pairs of inverse points also are obtained.

For the equality of the ratios $OP : OP'$, $OQ : OQ'$ necessitates the equality of the rectangles $OP \cdot OQ'$, $OQ \cdot OP'$.

§ 6. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the system of points similar to a given system A, B, C, ...

Figure 41.

Join O to A, B, C, ...; and on OA, OB, OC, ... find A′, B′, C′, ... such that
$$OA : OA' = OB : OB' = OC : OC' = \dots = r : r'.$$

Join PP′, QQ′, and let them intersect at O. Then O is the centre of inversion.

This follows from § 1′.

Case 2. When the four points are collinear.

FIRST METHOD.

Figure 38′.

Take any point M not collinear with the four points, and join PM, QM.

Through P′ describe a circle passing through P, M; through Q′ describe a circle passing through Q, M; and let these circles intersect at M′.

MM′ will intersect PP′ at O, the centre of inversion.

For $$OP \cdot OP' = OM \cdot OM', \\ = OQ \cdot OQ'.$$

SECOND METHOD.

Figure 39′.

Take any point M not collinear with the four points, and join PM, QM.

Through Q′ draw a straight line parallel to PM; through P′ draw a straight line parallel to QM; and let these straight lines intersect at M′.

MM′ will intersect PP′ at O, the centre of inversion.

For $$OP : OQ' = OM : OM', \\ = OQ : OP';$$

therefore $$OP \cdot OP' = OQ \cdot OQ'.$$

§ 5′. It will be seen from the two preceding methods of solution that, when two pairs of inverse points happen to be collinear, two pairs of similar points also are obtained.

For the equality of the rectangles OP·OP′, OQ·OQ′ necessitates the equality of the ratios OP : OQ′, OQ : OP′.

§ 6′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$, to find the system of points inverse to a given system A, B, C, ...

Figure 41′.

Join O to A, B, C, ... ; and on OA, OB, OC, ... find A′, B′, C′, ... such that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \ldots = r \cdot r'.$$

§ 7. If two systems of points be similar, the straight line joining any pair of points in the one is parallel to the straight line joining the corresponding pair in the other.

This follows from § 2.

§ 8. If two systems of points be similar, and three points of the first system be collinear, the three corresponding points of the second system will also be collinear.

Figure 40.

Let the system A, B, C be similar to the system A′, B′, C′, and let A, B, C be collinear.

Since, by § 7, A′B′ is parallel to AB,

therefore $\angle OB'A' = \angle OBA.$

Similarly $\angle OB'C' = \angle OBC;$

therefore $\angle OB'A' + \angle OB'C' = \angle OBA + \angle OBC,$

$= 2$ right angles;

therefore A′, B′, C′ are collinear.

Hence the curve similar to a straight line is a straight line.

§ 9. If two systems of points be similar, with respect to a centre of similitude O and a ratio of similitude $r : r'$, then for every two points A, B and the points A′, B′ similar to them

$$A'B' : AB = r' : r.$$

Figure 41.

Since A′B′ is parallel to AB,

therefore triangles OA′B′, OAB are equiangular;

therefore $A'B' : AB = OA' : OA,$

$= r' : r.$

If p', p be the perpendiculars from O on A′B′ and AB, it also follows from the equiangularity of the triangles OA′B′, OAB that

$$A'B' : AB = p' : p,$$

§ 10. If two systems of points be similar, then for every three points A, B, C and the points A′, B′, C′ similar to them

$$B'C' : C'A' : A'B' = BC : CA : AB.$$

§ 7′. If two systems of points be inverse, the straight line joining any pair of points in the one is antiparallel to the straight line joining the corresponding pair in the other.

This follows from § 2′.

§ 8′. If two systems of points be inverse, and three points of the first system be collinear, the three corresponding points of the second system will not in general be collinear.

Figure 40′.

Let the system A, B, C be inverse to the system A′, B′, C′, and let A, B, C be collinear.

Since, by § 7′, A′B′ is antiparallel to AB,

therefore $\angle OA'B' = \angle OBA.$

Similarly $\angle OC'B' = \angle OBC$

therefore $\angle OA'B' + \angle OC'B' = \angle OBA + \angle OBC,$

$= 2$ right angles;

which is impossible, if A′, B′, C′ be collinear.

Hence the curve inverse to a straight line is not in general a straight line.

§ 9′. If two systems of points be inverse, with respect to a centre of inversion O and a rectangle of inversion $r{\cdot}r'$, then for every two points A, B and the points A′, B′ inverse to them

$$A'B' : AB = r{\cdot}r' : OA{\cdot}OB.$$

Figure 41′.

Since A′B′ is antiparallel to AB with respect to angle AOB, therefore triangles OA′B′, OAB are equiangular;

therefore $A'B' : AB = OA' : OB = OA{\cdot}OA' : OA{\cdot}OB,$

$= r{\cdot}r' : OA{\cdot}OB.$

If p', p be the perpendiculars from O on A′B′ and AB, it also follows from the equiangularity of the triangles OA′B′, OAB that

$$A'B' : AB = p' : p.$$

§ 10′. If two systems of points be inverse, then for every three points A, B, C and the points A′, B′, C′ inverse to them

$$B'C' : C'A' : A'B' = OA{\cdot}BC : OB{\cdot}CA : OC{\cdot}AB.$$

Figure 41.

For $$B'C' = BO.\frac{r'}{r},\ C'A' = CA.\frac{r'}{r};$$

therefore $$\frac{B'C'}{C'A'} = BO.\frac{r'}{r}/CA.\frac{r'}{r},$$
$$= BC/CA.$$

§ 11. If two systems of points be similar, then for every four points A, B, C, D and the points A′, B′, C′, D′ similar to them

$$B'C' \cdot A'D' : C'A' \cdot B'D' : A'B' \cdot C'D' =$$
$$BC \cdot AD : CA \cdot BD : AB \cdot CD.$$

Figure 41.

For $$B'C' \cdot A'D' = BC.\frac{r'}{r}.AD.\frac{r'}{r},$$

and $$C'A' \cdot B'D' = CA.\frac{r'}{r}.BD.\frac{r'}{r};$$

therefore $B'C' \cdot A'D' : C'A' \cdot B'D' = BC \cdot AD : CA \cdot BD.$

§ 12. Every straight line passing through a centre of similitude cuts two similar curves C, C′ at similar points.

This follows from § 1.

§ 13. Every straight line passing through a centre of similitude and touching a curve C will also touch the similar curve C′.

Figure 42.

Let a straight line through the centre of similitude O cut C at P and Q, then it will cut C′ at P′ and Q′ the points similar to P and Q. Now, when the points P and Q move up to each other and ultimately coincide, that is, when the straight line touches C, the points P′ and Q′ will move up to each other and ultimately coincide, that is, the straight line will touch C′.

§ 14. If on two similar curves C, C′ similar points P, P′ be taken, the tangents at P, P′ make equal angles with OPP′.

Figure 43.

On C take any point Q near to P, and on C′ find the point Q′ similar to Q. Draw the secants QPR, Q′P′R′.

Figure 41′.

For $$B'C' = BC.\frac{rr'}{OB \cdot OC},\quad C'A' = CA.\frac{r.r'}{OC \cdot OA};$$

therefore $$\frac{B'C'}{C'A'} = BC.\frac{rr'}{OB \cdot OC}/CA.\frac{rr'}{OC \cdot OA},$$
$$= OA \cdot BC/OB \cdot CA.$$

§ 11′. If two systems of points be inverse, then for every four points A, B, C, D and the points A′, B′, C′, D′ inverse to them
$$B'C' \cdot A'D' : C'A' \cdot B'D' : A'B' \cdot C'D' =$$
$$BC \cdot AD \;:\; CA \cdot BD \;:\; AB \cdot CD.$$

Figure 41′.

For $$B'C' \cdot A'D' = BC.\frac{rr'}{OB \cdot OC}.AD.\frac{rr'}{OA \cdot OD},$$

and $$C'A' \cdot B'D' = CA.\frac{rr'}{OC \cdot OA}.BD.\frac{rr'}{OB \cdot OD};$$

therefore $$B'C' \cdot A'D' : C'A' \cdot B'D' = BC \cdot AD : CA \cdot BD.$$

§ 12′. Every straight line passing through a centre of inversion cuts two inverse curves C, C′ at inverse points.

This follows from § 1′.

§ 13′. Every straight line passing through a centre of inversion and touching a curve C will also touch the inverse curve C′.

Figure 42′.

Let a straight line through the centre of inversion O cut C at P and Q, then it will cut C′ at P′ and Q′ the points inverse to P and Q. Now, when the points P and Q move up to each other and ultimately coincide, that is, when the straight line touches C, the points P′ and Q′ will move up to each other and ultimately coincide, that is, the straight line will touch C′.

§ 14′. If on two inverse curves C, C′ inverse points P, P′ be taken, the tangents at P, P′ make supplementary angles with OPP′.

Figure 43′.

On C take any point Q near to P, and on C′ find the point Q′ inverse to Q. Draw the secants QPR, Q′P′R′.

Then the secants QR, Q′R′ are parallel;

therefore $\angle OQP = \angle OQ'P'$.

Now, when Q moves to coincidence with P, that is, when the secant QR becomes the tangent PT, Q′ moves to coincidence with P′, that is, the secant Q′R′ becomes the tangent P′T′.

Also when Q moves to coincidence with P, $\angle OQP$ becomes $\angle OPS$; and when Q′ moves to coincidence with P′, $\angle OQ'P'$ becomes $\angle OP'S'$;

therefore $\angle OPS = \angle OP'S'$;

therefore $\angle OPT = \angle OP'T'$.

§ 15. If two curves intersect each other at any angle, the curves similar to them intersect each other at the same angle.

Figure 44.

Let the two curves C and D intersect each other at P; then C′ and D′ the curves similar to them will intersect each other at P′ the point similar to P.

Draw PT, PU tangents to C and D; and P′T′, P′U′ tangents to C′ and D′; and let O be the centre of similitude.

Then $\angle OPT = \angle OP'T'$

and $\angle OPU = \angle OP'U'$;

therefore $\angle OPT - \angle OPU = \angle OP'T' - \angle OP'U'$;

therefore $\angle TPU = \angle T'P'U'$.

Hence, if two curves touch each other at any point P, the curves similar to them will touch each other at the similar point P′.

§ 16. If two curves C′, C″ be both similar to a curve C with respect to the same centre of similitude O, then C′, C″ are similar to each other.

Let $r : r'$ and $r : r''$ be the two ratios of similitude.

Take any point P on C, and the similar points P′, P″ on C′ and C″.

Then $OP : OP' = r : r'$,

$OP : OP'' = r : r''$;

therefore $$\frac{OP'}{OP} \Big/ \frac{OP''}{OP} = \frac{r'}{r} \Big/ \frac{r''}{r};$$

therefore $$OP' : OP'' = \frac{r'}{r} : \frac{r''}{r}$$

$= \text{a constant ratio.}$

Hence C′ is similar to C″.

Then the secants QR, Q′R′ are antiparallel;

therefore $\angle OPQ = \angle OQ'P'$.

Now, when Q moves to coincidence with P, that is, when the secant QR becomes the tangent PT, Q′ moves to coincidence with P′, that is, the secant Q′R′ becomes the tangent P′T′.

Also when Q moves to coincidence with P, $\angle OPQ$ becomes $\angle OPT$; and when Q′ moves to coincidence with P′, $\angle OQ'P'$ becomes $\angle OP'S'$;

therefore $\angle OPT = \angle OP'S'$;

therefore $\angle OPT =$ supplement of $\angle OP'T'$.

§ 15′. If two curves intersect each other at any angle, the curves inverse to them intersect each other at the same angle.

Figure 44′.

Let the two curves C and D intersect each other at P; then C′ and D′ the curves inverse to them will intersect each other at P′ the point inverse to P.

Draw PT, PU tangents to C and D; and P′T′, P′U′ tangents to C′ and D′; and let O be the centre of inversion.

Then $\angle OPT =$ supplement of $\angle OP'T'$

and $\angle OPU =$ supplement of $\angle OP'U'$;

therefore $\angle OPT - \angle OPU =$ supplement of $\angle OP'T'$ − supplement of $\angle OP'U'$;

therefore $\angle TPU = \angle T'P'U'$.

Hence, if two curves touch each other at any point P, the curves inverse to them will touch each other at the inverse point P′.

§ 16′. If two curves C′, C″ be both inverse to a curve C with respect to the same centre of inversion O, then C′, C″ are similar to each other.

Let $r \cdot r'$ and $r \cdot r''$ be the two rectangles of inversion.

Take any point P on C, and the inverse points P′, P″ on C′ and C″.

Then $OP \cdot OP' = r \cdot r'$,

$OP \cdot OP'' = r \cdot r''$;

therefore $OP \cdot OP' / OP \cdot OP'' = r.r' / r.r''$;

therefore $OP' : OP'' = r \cdot r' : r \cdot r''$

$=$ a constant ratio.

Hence C′ is similar to C″.

SIMILITUDE.

§ 17. Given a centre of similitude O and a ratio of similitude $r : r'$, to find the curve similar to a given straight line PQ.

Case 1. When PQ passes through O.

Figure 45.

Take any point P in PQ, and find in OP the point P′ such that $OP : OP' = r : r'$. P and P′ will be on the same side of O or on opposite sides of O according as $r : r'$ is positive or negative.

Since the ratio OP : OP′ is fixed, as OP increases OP′ will also increase; and as OP diminishes OP′ will also diminish. In other words, as P moves farther and farther from O, P′ will also move farther and farther from O; as P moves nearer and nearer to O, P′ will also move nearer and nearer to O. And consequently when P is infinitely distant from O, P′ will also be infinitely distant from O; when P coincides with O, P′ will also coincide with O. Hence when P describes from right to left or from left to right the straight line PQ, P′ will also describe from right to left or from left to right the same straight line.

Case 2. When PQ does not pass through O.

Figure 46.

Through O draw OP perpendicular to PQ, and find in OP the point P′ such that $OP : OP' = r : r'$; through P′ draw P′Q′ perpendicular to OP′.

This perpendicular is the curve similar to PQ.

Take any point Q in PQ, join OQ, and let it meet the perpendicular in Q′.

Since angles OPQ, OP′Q′ are right,
therefore PQ and P′Q′ are parallel;
therefore $$OQ : OQ' = OP : OP' = r : r'.$$

Hence Q′ is the point similar to Q, and as Q was any point whatever in PQ, therefore all the points in PQ have the points similar to them situated in P′Q′; that is, the straight line P′Q′ is the curve similar to the given straight line PQ.

§ 18. If the straight line P′Q′ is similar to the straight line PQ with respect to a given centre of similitude O, the reciprocal relation also holds good, namely, that the straight line PQ is similar to the straight line P′Q′ with respect to the same centre of similitude.

§ 17′. Given a centre of inversion O and a rectangle of inversion $r \cdot r'$, to find the curve inverse to a given straight line PQ.

Case 1. When PQ passes through O.

Figure 45′.

Take any point P in PQ, and find in OP the point P′ such that $OP \cdot OP' = r \cdot r'$. P and P′ will be on the same side of O or on opposite sides of O according as $r \cdot r'$ is positive or negative.

Since the rectangle OP·OP′ is fixed, as OP increases OP′ will diminish; and as OP diminishes OP′ will increase. In other words, as P moves farther and farther from O, P′ will move nearer and nearer to O; as P moves nearer and nearer to O, P′ will move farther and farther from O. And consequently when P is infinitely distant from O, P′ will coincide with O; when P coincides with O, P′ will be infinitely distant from O. Hence when P describes from right to left or from left to right the straight line PQ, P′ will also describe from left to right or from right to left the same straight line.

Case 2. When PQ does not pass through O.

Figure 46′.

Through O draw OP perpendicular to PQ, and find in OP the point P′ such that $OP \cdot OP' = r \cdot r'$; on OP′ as diameter describe a circle.

This circle is the curve inverse to PQ.

Take any point Q in PQ, join OQ, and let it meet the circle in Q′; and join P′Q′.

Since angles OPQ, OQ′P′ are right,

therefore PQ and P′Q′ are antiparallel with respect to angle POQ;

therefore $$OQ \cdot OQ' = OP \cdot OP' = r \cdot r'.$$

Hence Q′ is the point inverse to Q, and as Q was any point whatever in PQ, therefore all the points in PQ have the points inverse to them situated on the circle OP′Q′; that is, the circle OP′Q′ is the curve inverse to the given straight line PQ.

§ 18′. If the circle OP′Q′ is inverse to the straight line PQ with respect to a given centre of inversion O, the reciprocal relation also holds good, namely, that the straight line PQ is inverse to the circle OP′Q′ with respect to the same centre of inversion.

§ 19. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the curve similar to a circle.

Case 1. When the circle passes through O.

Figure 47.

Let OPQ be the given circle.

Through O draw the diameter OP, and in OP find the point P' such that $OP : OP' = r : r'$; on OP' as diameter describe the circle OP'Q'.

This circle is the curve similar to OPQ.

Take any point Q in OPQ, join OQ, and let it meet the circle OP'Q' at Q'. Join PQ, P'Q'.

Since angles OQP and OQ'P' are right,
therefore PQ and P'Q' are parallel;
therefore $$OQ : OQ' = OP : OP' = r : r'.$$

Hence Q' is the point similar to Q, and as Q was any point whatever in OPQ, therefore all the points in OPQ have the points similar to them situated in OP'Q'; that is, the circle OP'Q' is the curve similar to the given circle OPQ.

Case 2. When the circle does not pass through O.

Figure 48.

Let C be the centre of the given circle.

Take any point P on the circle C, join OP, and in OP find P' similar to P. Join PC, and at P' make angle OP'C' equal to angle OPC, and let P'C' meet OC at C'. With C' as centre and C'P' as radius describe a circle.

This circle will be similar to the circle C.

Since $$\angle OPC = \angle OP'C',$$
therefore PC and P'C' are parallel;
therefore $$OP : OP' = CP : C'P'.$$

Hence $$C'P' = CP.\frac{OP'}{OP}$$
$$= \text{a constant},$$
since CP is a constant length, and $OP : OP'$ is a constant ratio.

INVERSION.

§ 19′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$, to find the curve inverse to a circle.

Case 1. When the circle passes through O.

Figure 47′.

Let OPQ be the given circle.

Through O draw the diameter OP, and in OP find the point P′ such that $OP \cdot OP' = r \cdot r'$; at P′ draw P′Q′ perpendicular to OP′.

This perpendicular is the curve inverse to OPQ.

Take any point Q in OPQ, join OQ, and let it meet the perpendicular P′Q′ at Q′. Join PQ.

Since angles OQP and OP′Q′ are right,
therefore PQ and P′Q′ are antiparallel with respect to angle POQ;
therefore $$OQ \cdot OQ' = OP \cdot OP' = r \cdot r'.$$

Hence Q′ is the point inverse to Q, and as Q was any point whatever in OPQ, therefore all the points in OPQ have the points inverse to them situated in P′Q′; that is, the straight line P′Q′ is the curve inverse to the given circle OPQ.

Case 2. When the circle does not pass through O.

Figure 48′.

Let C be the centre of the given circle.

Take any point P on the circle C, join OP, and in OP find P′ inverse to P. Join PC, and at P′ make angle OP′C′ equal to the supplement of angle OPC, and let P′C′ meet OC at C′. With C′ as centre and C′P′ as radius describe a circle.

This circle will be inverse to the circle C.

Let OP meet the circle C again at Q, and join CQ.

Since supplement of $\angle OPC = \angle OP'C'$,
therefore $$\angle OQC = \angle OP'C';$$
therefore QC and P′C′ are parallel;
therefore $$OQ : OP' = CQ : C'P'.$$

Hence $$C'P' = CQ . \frac{OP'}{OQ} = CQ . \frac{OP \cdot OP'}{OP \cdot OQ}$$
$$= \text{a constant,}$$
since CQ is a constant length, and $OP \cdot OP'$, $OP \cdot OQ$ are constant rectangles.

Again $\qquad OP : OP' = OC : OC'$;

therefore $\qquad OC' = OC.\dfrac{OP'}{OP}$

$\qquad\qquad = \text{a constant.}$

therefore C′ is a fixed point.

Since C′ is a fixed point, and C′P′ is of constant length, therefore the locus of P′ is the circle C′.

§ 20. If the circle C′ is similar to the circle C with respect to a given centre of similitude, the reciprocal relation also holds good, namely, that C is similar to C′ with respect to the same centre of similitude.

§ 21. Given two circles C, C′ (whose radii are c, c') similar to each other, to find the centre of similitude.

Figure 48.

It will be seen from the construction and the reasoning in § 19, Case 2, that two circles have two centres of similitude, external or internal, according as their ratio of similitude is positive or negative.

Since $\qquad OC : OC' = CP : C'P',$

$\qquad\qquad = c : c',$

these centres of similitude are found by dividing CC′, the distance between the centres of the two circles, externally or internally, in the ratio of the radii.

§ 22. Given two circles C, C′ (whose radii are c, c') similar to each other, to find the ratio of similitude.

Figure 48.

Find O the external or internal centre of similitude by dividing CC′ externally or internally in the ratio $c : c'$; draw any straight line OPQ cutting the circles C, C′ in the pairs of similar points P and P′, Q and Q′; and join CP, C′P′.

Then $\qquad c : c' = CP : C'P',$

$\qquad\qquad = OP : OP'$;

therefore the ratio of similitude of two circles similar to each other is the ratio of their radii.

Again $\qquad OQ : OP' = OC : OC'$;

therefore $\qquad OC' = OC.\dfrac{OP'}{OQ} = OC\cdot\dfrac{OP\cdot OP'}{OP\cdot OQ}$

$= \text{a constant}$;

therefore C′ is a fixed point.

Since C′ is a fixed point, and C′P′ is of constant length, therefore the locus of P′ is the circle C′.

§ 20′. If the circle C′ is inverse to the circle C with respect to a given centre of inversion, the reciprocal relation also holds good, namely, that C is inverse to C′ with respect to the same centre of inversion.

§ 21′. Given two circles C, C′ (whose radii are c, c') inverse to each other, to find the centre of inversion.

Figure 48′.

It will be seen from the construction and the reasoning in § 19′, Case 2, that two circles have two centres of inversion, external or internal, according as their rectangle of inversion is positive or negative.

Since $\qquad OC : OC' = CQ : C'P'$,

$= c : c'$,

these centres of inversion are found by dividing CC′, the distance between the centres of the two circles, externally or internally, in the ratio of the radii.

§ 22′. Given two circles C, C′ (whose radii are c, c') inverse to each other, to find the rectangle of inversion.

Figure 48′.

Find O the external or internal centre of inversion by dividing CC′ externally or internally in the ratio $c : c'$; draw any straight line OPQ cutting the circles C, C′ in the pairs of inverse points P and P′, Q and Q′; and join CQ, C′P′.

Then $\qquad c : c' = CQ : C'P'$,

$= OQ : OP' = OP\cdot OQ : OP\cdot OP'$;

therefore the rectangle of inversion of two circles inverse to each other is a fourth proportional to c, c', and OP· OQ, the potency of O with respect to the circle C.

§ 23. A circle is similar to itself with respect to any centre of similitude O, when the ratio of similitude is unity.

For if P be any point on the circle,

since $$\mathrm{OP} : \mathrm{OP}' = 1,$$

therefore P′ is the same point as P.

Hence as P describes clockwise or counterclockwise the circumference of the given circle, P′ describes clockwise or counterclockwise the same circumference.

§ 24. If two circles be similar to each other their centres are similar points.

Figure 49.

Let C, C′ be two circles similar to each other, O their centre of similitude, and let OT touch the circle C at T. Then, by § 13, OT will touch the circle C′ at T′ the point similar to T. Join CT, C′T′.

Now, if the centre C be considered a point in the figure C, the point similar to it will, by § 12, be situated on OC. It will also, by § 7, be situated on a straight line through T′ parallel to TC. But since ∠ OTC and ∠ OT′C′ are right, therefore T′C′ is parallel to TC; therefore C′ is the point similar to C.

§ 25. The property of § 7 as applied to the circle may be enunciated thus:

If two circles be similar to each other the chord joining any two points of the one intersects the chord joining the two similar points of the other on the straight line at infinity.

The property of § 14 thus:

If two circles be similar to each other the tangents at two similar points of them intersect on the straight line at infinity.

§ 23′. A circle is inverse to itself with respect to any centre of inversion O, when the rectangle of inversion is the potency of O with respect to the given circle.

For if P be any point on the circle, since $OP \cdot OP'$ = the potency of O with respect to the given circle, therefore P′ is the point where OP cuts the given circle a second time. Hence as P describes clockwise or counterclockwise the circumference of the given circle, P′ describes counterclockwise or clockwise the same circumference.

§ 24′. If two circles be inverse to each other their centres are not inverse points.

Figure 49′.

Let C, C′ be two circles inverse to each other, O their centre of inversion, and let OT touch the circle C at T. Then, by § 13′, OT will touch the circle C′ at T′ the point inverse to T. Join CT, C′T′.

Now, if the centre C be considered a point in the figure C, the point inverse to it will, by § 12, be situated on OC. It will also, by § 7′, be situated on a straight line through T′ antiparallel to TC. But since ∠ OTC and ∠ OT′C′ are right, therefore T′C′ is parallel to TC; therefore C′ is not the point inverse to C.

§ 25′. The property of § 7′ as applied to the circle may be enunciated thus:

If two circles be inverse to each other the chord joining any two points of the one intersects the chord joining the two inverse points of the other on the radical axis of the two circles.

The property of § 14′ thus:

If two circles be inverse to each other the tangents at two inverse points of them intersect on the radical axis of the two circles.

Eighth Meeting, June 8th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

A Construction for the Brocard Points.

By R. E. ALLARDICE, M.A.

The following note may be considered as an addendum to the paper by me on pp. 42-47 of this volume of the *Proceedings.* In that paper it is shown how to inscribe in a triangle ABC, a triangle DEF, such that the perpendiculars to the sides of ABC, drawn through the points D, E, F, shall be concurrent in a point P. This is done by constructing on each of the sides of ABC a triangle similar to DEF; then O the point of concurrence of the three lines joining the vertices of ABC to the vertices of these triangles is the point "inverse" to P. The question, then, naturally arises, What must be the shape of the triangle DEF in order that the point P may be one of the Brocard points, and, as a consequence, O the other one? and the answer is easily seen to be that DEF must be similar to ABC. Hence the following construction:—

On the sides BC, CA, make the triangles CEB, CAF, similar to ABC; then the point of concurrence of AE and BF is one of the Brocard points. The point E may be obtained by drawing BE parallel to AC and making BE a third proportional to AC and CB; and a similar construction may be given for the point F.

If, instead of making CEB and CAF similar to ABC (where the correspondence of vertices is indicated by the order in which the triangle is named), we make BCE and FCA similar to ABC, we obtain the second Brocard point.

ON THE TEACHING OF ARITHMETIC.

REPORT drawn up by the Committee appointed 13th January 1888, and approved by the Society, 8th June 1888.

---

1. The following report is not a syllabus of arithmetical teaching, far less a formal treatise, but an attempt to reduce to some order the miscellaneous suggestions at the disposal of the committee.* These suggestions are roughly classified into seven groups. Arithmetic includes both a science and an art, or a theory and applications of it to the wants of life. Of the seven divisions in this report the first four have mainly to do with the art, the next two with the science of arithmetic (the division is not observed with thoroughgoing strictness), while the seventh division contains a few general hints for teaching. These divisions may be briefly styled :—

(1.) Terms, (2.) Measures and Constants, (3.) Processes, (4.) Computation, (5.) Theory, (6.) Links, or ideas leading to higher mathematics, and (7.) General Hints.

I. Terms.

2. The meaning of certain terms relating to quantity falls to be taught under Arithmetic.

(*a*) Such as occur frequently, as

Sum, difference, product, quotient, fraction, ratio, reciprocal, prime, odd and even numbers, measure, multiple, power, root, &c.

Per cent., interest, discount, stock, area, volume, average, &c.

(*b*) Such as occur less frequently, as

Index, logarithm, surd, commensurable, &c.

Specific gravity, velocity, acceleration, work, horse-power.

Of these (*a*) should be thoroughly mastered, while (*b*) should be left for advanced pupils, or explained as they occur in problems.

To facilitate the mastery of several terms in (*a*), the pupil should be made to use, instead of the general word "answer," the particular term appropriate to the case, as sum, difference, G.C.M., price per yard, &c.

---

* The Committee was as follows: Dr J. S. Mackay, Messrs A. J. G. Barclay, A. Y. Fraser (Convener), W. J. Macdonald, and Robert Robertson (who was prevented by domestic bereavement from attending the meetings). At the later meetings Mr R. E. Allardice was present by invitation of the Committee. The suggestions above referred to were made at the December and January meetings, and also by members in communications to the Secretary.

II. Measures and Constants.

3. Of the tables of measures to be committed to memory, the following are recommended as necessary and sufficient :—

Money.

12 pence (*d.*) = 1 shilling (*s.* or /)
20 shillings = 1 *pound* (£.)

Weight.

16 ounces (*oz.*) = 1 *pound* (*lb.*)
112 pounds = 1 hundredweight (*cwt.*)
20 hundredweights = 1 ton.

[7000 grains (*gr.*) = 1 pound. 14 pounds = 1 stone (*st.*)]

Length.

12 inches (*in.*) = 1 foot (*ft.*)
3 feet = 1 *yard* (*yd.*)
1760 yards = 1 mile (*ml.*)

[22 yards = 1 chain (*ch.*) = 100 links (*lk.*)]

Surface.

144 or 12×12 square inches (*sq. in.*) = 1 square foot (*sq. ft.*)
9 or 3×3 square feet = 1 square yard (*sq. yd.*)
4840 square yards = 1 acre (*ac.*)
640 acres = 1 square mile (*sq. ml.*)

[40 poles (*pl.*) = 1 rood (*ro.*) 4 roods = 1 acre = 10 square chains.]

Solidity.

1728 or $12 \times 12 \times 12$ cubic inches (*c. in.*) = 1 cubic foot (*c. ft.*)
27 or $3 \times 3 \times 3$ cubic feet = 1 cubic yard (*c. yd.*)

Liquid Measure.

4 gills = 1 pint (*pt.*)
2 pints = 1 quart (*qt.*)
4 quarts = 1 *gallon* (*gal.*)

Dry Measure.

4 pecks (*pk.*) = 1 bushel (*bus.*)
8 bushels = 1 quarter (*qr.*)

[2 gallons = 1 peck.]

Time.

60 seconds (*sec.*) = 1 minute (*min.*)
60 minutes = 1 hour (*hr.*)
24 hours = 1 *day.*
365 days = 1 common year (*yr.*)
366 days = 1 leap year.

In the foregoing tables the standard units are printed in italics. Fractions are not used.

Other tables to be retained for reference merely are Apothecary's weight, Troy weight, angular measure, foreign measures, &c.

4. Constants of frequent occurrence should be remembered, such as the circumference of a circle is 3·1416 times its diameter nearly, the circumference of the Earth is 25000 miles nearly, 1 cubic foot of water weighs 1000 ounces nearly, &c.

Others of less frequent occurrence should be tabulated for reference, as, specific gravities of common substances, iron, lead, silver, gold, mercury, &c.; one gallon of water weighs 10 lb., and occupies 277·274 cubic inches nearly; fathom, knot; speed of light, of sound, of falling bodies, atmosphere as a unit of pressure, &c.

III. Processes.

5. It should be clearly understood that the fundamental operations of arithmetic are few. From addition and subtraction are developed multiplication and division, involution and evolution; and the numerous "Rules" in text-books are for the most part classified lists of examples of the application of these operations. As a matter of fact, the fundamental laws of arithmetic and algebra together are only three in number (see § 28).

6. *Decimals.*—It should be shown that the decimal notation for fractions (that is "Decimal Fractions") is merely an extension of the ordinary notation for integers.

7. *Metric System.*—The metric system should be taught, even if the time at the teacher's disposal is so small as to compel him to restrict his treatment to a single measure, for example, length. The principle of the system is too valuable to be passed over.

8. *Practice.*—The old rule of Practice should be omitted. In the cases where a simple fraction will suffice, retain such exercises as, to find the value of a number of articles whose price is a simple fraction of a pound, of a shilling, &c. These exercises can be introduced as applications of fractions.

9. *The Unitary Method and Proportion.*—Opinion is by no means unanimous as to the relative values of these two processes. There seems to be no doubt that the unitary method can be used with intelligence at an earlier age than that of proportion, but it should not supplant proportion throughout. The ideas involved in proportion are of more general importance than anything in the unitary method.

10. *Evolution.*—The extraction of square root should be taught as a part of arithmetic, because the need for it arises so frequently; the extraction of cube root should be treated as a part of algebra.

The following are useful approximations, especially the first two.

$$\begin{array}{lllll} \sqrt{2} = \sqrt{(50/25)} = \sqrt{(49/25)} & \text{nearly} & = 7/5 & \text{nearly.} \\ \sqrt{3} = \sqrt{(48/16)} = \sqrt{(49/16)} & ,, & = 7/4 & ,, \\ \sqrt{5} = \sqrt{(80/16)} = \sqrt{(81/16)} & ,, & = 9/4 & ,, \\ \sqrt{7} = \sqrt{(63/9)} = \sqrt{(64/9)} & ,, & = 8/3 & ,, \\ \sqrt{11} = \sqrt{(99/9)} = \sqrt{(100/9)} & ,, & = 10/3 & ,, \end{array}$$

[*Note.* $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{9}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12} = 15/6$, $16/6$, $17/6$, $18/6$, $19/6$, $20/6$, $21/6$.]

IV. Computation.

11. Systematic endeavour should be made to develope in the pupil the power of accurate and rapid computation. This accomplishment can be attained by continued practice. It requires no severer mental strain than does the acquirement of dexterity in any mechanical exercise.

12. Beginners should be taught to compute in such a way that there may, if possible, be nothing to unlearn. "The act of addition must be made in the mind without assistance; you must not permit yourself to say, 4 and 7 are 11, 11 and 7 are 18, &c., but only 4, 11, 18, &c." "Learn the multiplication table so well as to name the product the instant the factors are seen; that is, until 8 and 7, or 7 and 8, suggest 56 at once, without the necessity of saying '7 times 8 are 56.'" (De Morgan's *Arithmetic*, 5th edition, pp. 162, 163.)

13. The addition table should be committed to memory as strictly as the multiplication table—both having been first of all constructed by the pupil, with the aid of counters or the like.

14. There should be systematic practice at exercises containing all the typical difficulties (compare the pianist's practice to acquire

technical dexterity.) For exercises of this kind see De Morgan (1) *Arithmetic*, 5th ed., appendix, (2) *Supplement to Penny Cyclopædia*, article *Computation*, (3) *British Almanac and Companion* for 1844.

15. As a further aid to rapid computation, the pupil should be exercised in certain labour-saving devices, namely, (1) methods to meet particular frequently-occurring cases (see § 16 following), and (2) the general abbreviated methods of working to produce results to any required degree of approximation (see §§ 18, 19).

16. *Short and easy methods.*—It is not suggested that all the methods in the following list (which does not pretend to be exhaustive) should be systematically taught.

Addition.—*Example.*—In adding two numbers, the first or second may be increased, and the second or first diminished to the same extent.

As, $57 + 39 = 60 + 36.$
$\text{or} = 56 + 40.$

Or again, $57 + 39 = 57 + 30 + 9.$
$\text{or} = 50 + 39 + 7.$

Subtraction.—*Example.*—In subtracting two numbers, both may be equally increased or diminished.

As, $57 - 39 = 58 - 40$
$346 - 328 = 46 - 28$

The following is frequently useful:

$$100 - 58{\cdot}2476 = 41{\cdot}7524.$$

Subtract, 5 from 9, 8 from 9, 2 from 9, &c., 6 from 10.

Multiplication.—When the multiplier ends with 5, in general double it, and halve the answer, or, when the multiplicand is even, halve the multiplicand before multiplying.

To multiply by $5,\ 5^2,\ 5^3,\ 5^4$, &c.
multiply by $10,\ 10^2,\ 10^3,\ 10^4$, &c.
and divide by $2,\ 2^2,\ 2^3,\ 2^4$, &c.

To multiply by 9, 99, 999, &c.
multiply by 10, 100, 1000, &c.
and subtract the multiplicand. No working need be written down, only the answer.

To multiply by 11, 111, 1111, &c. If by 11, suppose a zero to be placed to right and left of the multiplicand. Beginning from right, add the digits in successive pairs. If by 111, suppose two zeros so placed, and add in successive threes, &c.

To multiply by 98, 97, 96, &c., multiply by 100, and subtract 2, 3, 4, &c., times the multiplicand. Similarly for 998, 997, &c.

To multiply by 42, 63, 189, &c., use the first line of the partial product instead of the multiplicand to get the second line.

To square a number of two digits. The identity

$a^2 = (a-b)(a+b)+b^2$, gives for example $58^2 = 56 \times 60 + 2^2$.

Division.—When the divisor ends in 5, in general double both divisor and dividend.

To divide by	5, 5^2, 5^3, 5^4, &c.
multiply by	2, 2^2, 2^3, 2^4, &c.
and divide by	10, 10^2, 10^3, 10^4, &c.

To find the value of 12 articles, given the price of one. Consider the pence in the price as shillings.

To find the value of 20 articles, given the price of one. Consider the shillings in the price as pounds.

To find the value of 100 articles, given the price of one. For every farthing take as many pence and twice as many shillings.

To find the price of 112 pounds, given the price of one pound. Add the price of 12 to the price of 100.

Percentage.—To find 5 % of a sum of money, take 1s. for every £1 in the sum. For $2\frac{1}{2}$ % take 6d. for every £1.

Five per cent. per annum is a penny per pound per month. Similarly for two and a half per cent.

17. Practice in the approximate methods should be kept up from the earliest acquaintance with the extended decimal notation. The young pupil, instead of being pushed through the weary and unprofitable mazes of circulating decimals, should rather be exercised in the metric system, in approximate addition, subtraction, multiplication, and division, and in learning to turn the decimal system to account in treating questions of British money.

18. The abbreviated methods of multiplication and division are now given in all good text books, and therefore need not be treated in detail here. The following recommendations are made.

a. In contracted multiplication the multiplier need not be written reversed. This procedure, though a useful and obvious expedient to the mature judgment, sometimes puzzles beginners. The pupil should be accustomed to multiply, beginning at the left as well

as at the right, and if the units, tens ... digits of the multiplier be placed under the units, tens ... digits of the multiplicand, the pupil easily learns to take the figures of the multiplier in *any order whatever*, and to place the product correctly. When this point has been reached, he may be allowed to carry on multiplication from the right in the usual way, and then by applying the principle involved in the statement that "a 3rd place decimal, multiplied by a 2nd place decimal, gives a 5th place decimal," he readily learns where he should begin to multiply in order to obtain the requisite approximation.

(*b*) In performing any long division the pupil should be required to get the successive remainders without writing down the partial products. This is quite easy, even without the special preparation recommended in § 14.

Thus the procedure to get the last remainder in the annexed example, stated in its fullest form for the sake of illustration, is:—

4 times 7, 28 and 9 (write 9 down) 37, carry 3; 4 times 5, 20, and 3 carried, 23 and 1 (write 1 down) 24, carry 2; 4 times 3, 12 and 2 carried, 14, and 2 (write 2 down) 16.

```
357)83757(234
      1235
       1647
        219
```

19. To express shillings, pence, and farthings as the decimal of a pound, note that

$$\begin{aligned} 2/ &= \cdot 1 \quad \text{of } £1. \\ 1/ &= \cdot 05 \quad \text{of } £1. \\ \tfrac{1}{4}\text{d.} &= \cdot 001 + 1/24 \text{ of } \cdot 001 \text{ of } £1. \end{aligned}$$

Hence the following rule, which gives the result correct to the thousandth of a £: Call the florins tenths, the odd shilling (if any) five hundredths; for the thousandths reduce the rest to farthings, adding 1 if there be more than 12, 2 if more than 36.

Examples: (1) £2 „ 7 „ $1\frac{3}{4}$

$$\begin{aligned} 3\text{fl.} &= \cdot 3 \\ 1\text{s.} &= \cdot 05 \\ 7\text{f.} &= \cdot 007 \end{aligned}$$

£2·357

(2) $£3 \,\shortparallel\, 16 \,\shortparallel\, 5\frac{1}{4}$

$$
\begin{aligned}
8\text{fl.} &= \cdot 8 \\
21\text{f.} &= \cdot 022
\end{aligned}
$$

$£3{\cdot}822.$

(3) $£4 \,\shortparallel\, 15 \,\shortparallel\, 10\frac{1}{4}$

$$
\begin{aligned}
7\text{fl.} &= \cdot 7 \\
1\text{s.} &= \cdot 05 \\
41\text{f.} &= \cdot 043
\end{aligned}
$$

$£4{\cdot}793.$

The inverse operation is as follows:—

(1)
$$
\begin{aligned}
£7{\cdot}293 &= £7{\cdot}2 \quad + \cdot 05 + \cdot 043. \\
&= £7 \,\shortparallel\, 4/ + 1/ \quad + 41\text{f.} \\
&= £7 \,\shortparallel\, 5 \,\shortparallel\, 10\tfrac{1}{4}.
\end{aligned}
$$

(2)
$$
\begin{aligned}
£4.567 &= £4{\cdot}5 + \cdot 05 + \cdot 017. \\
&= £4 \,\shortparallel\, 11 \,\shortparallel\, 4.
\end{aligned}
$$

Verifiable exercises on the above can be got at once from the ordinary examples of the compound rules. Turn a column of a compound addition sum into decimals, add the lines as decimals, convert the answer into £ s. d., and compare the result with that obtained in the usual way. Similarly with subtraction, multiplication, and division. The pupil will see how the absolute error increases in multiplication and diminishes in division, the relative error remaining the same.

20. *Logarithms.*—The last matter requiring discussion under the head of computation is the question of teaching logarithms.

It is recommended that a knowledge of logarithms be considered a necessary part of a complete arithmetical training. The use of logarithms with a sufficient amount of theory can be taught without the aid of advanced algebra.

V. Theory.

21. The necessity of developing the theory of arithmetic, in immediate connection with the practice of the special processes is universally acknowledged, but there is some difficulty in effecting this. A few suggestions are offered on points where a knowledge of theory is most frequently found wanting.

22. To recommend any text-books in use in this country would be invidious and perhaps useless, but such objection does not apply to foreign treatises. The two following may be mentioned as affording instances of the adaptation to arithmetic of theory which is usually left to algebra:—

(1.) *Traité Élémentaire d'Arithmétique*, par J. F. J. Kleyer [part 1 (75 centimes), part 2 (75 centimes), part 3 (1·10 francs)]. H. Dessain, Liége, rue Trappe, 7. 1885–6.

(2.) *Traité d'Arithmétique Élémentaire*, par L'Abbé E. Gelin (5 francs), Paris, Librairie du *Journal de Mathématiques Élémentaires*, rue des Écoles, 17. 1886.

23. *Scales of Notation.*—The theory of other scales of notation may be introduced to illustrate the decimal scale, if only to show the pupil that there is nothing essential in the selection of 10 as the radix of the ordinary scale. The Roman notation should be explained, and examples given in writing and reading it.

24. *Divisibility.*—The tests of divisibility of numbers by 2, 3, &c., should be given and proved.

25. *Greatest Common Measure.*—The theory of the method of finding the G. C. M. by reduction to primes, and of the ordinary method, should be explained.

This last may be done by means of the two following theorems, of which proofs may be given that are perfectly general in method, although particular numbers may be employed.

a. Every common measure of two numbers, say 36 and 28, is a measure of their difference 8 (and also of their sum 64).

Thus every common measure of 36 and 28 is a common measure of 28 and 8.

b. Conversely, every common measure of the less of two numbers 28 and their difference 8 is a measure of the greater of the numbers 36.

Thus any common measure of 28 and 8 is a common measure of 36 and 28.

From these two theorems it follows that the problem of finding the G. C. M. of the numbers 36 and 28 may be reduced to the simpler problem of finding the G. C. M. of the numbers 28 and 8, and this again to that of finding the G. C. M. of 20 and 8, and so on till the G. C. M. is found.

Moreover, instead of taking the difference of, say, 28 and 8, then the difference of 20 and 8, and again the difference of 12 and 8, we may take the difference of 28 and any multiple of 8, for instance, the difference of 28 and 8×3 or the difference of 28 and 8×4.

26. *Proportion.*—The following is given as a very brief outline, sufficient for arithmetic, of the theory of proportion. In the proofs letters are used, but they are meant to be given alongside of full numerical exemplification (see § 29).

Definition 1.—The ratio of one whole number to another is measured by the fraction which the one is of the other; and the ratio of one quantity to another is the ratio of the two whole numbers that express these quantities in terms of the same unit.

Definition 2.—When two ratios are equal, the four quantities form a proportion.

Theorem 1.—If the fourth term of a proportion be greater than the third, the second shall be greater than the first, if equal equal, and if less less.

Let $a : b = c : d$ be the proportion.

Because $a : b = c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

Now (1) if d be greater than c,

then the fraction $\frac{c}{d}$ is less than 1;

therefore the fraction $\frac{a}{b}$ is less than 1;

therefore b is greater than a.

(2) If d be equal to c,

then the fraction $\frac{c}{d}$ is equal to 1;

therefore the fraction $\frac{a}{b}$ is equal to 1;

therefore b is equal to a.

(3) If d be less than c,

then the fraction $\frac{c}{d}$ is greater than 1;

therefore the fraction $\frac{a}{b}$ is greater than 1;

therefore b is less than a.

Theorem 2.—In a proportion the product of the first and fourth terms is equal to the product of the second and third.

Let $a : b = c : d$ be the proportion.

Because $a : b = c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$;

therefore $\frac{a}{b} \times b \times d = \frac{c}{d} \times d \times b$;

therefore $a \times d = b \times c$.

Application—First.—Theorem 1 enables us to arrange in a proper order any four quantities among which a proportion is assumed to exist.

Example.—If 2 lb. of tea cost 5/, what will 6 lb. cost?

It is assumed that the ratio of 2 lb. to 6 lb. is equal to the ratio of the price of 2 lb. to the price of 6 lb.

Let x/ denote the unknown price, which it is usual to write as the fourth term of the proportion. Then the third term will be 5/, for a ratio can exist only between quantities of the same kind.

To ascertain whether 2 lb. or 6 lb. should occupy the second place, we inquire whether x/ is greater than 5/, equal to it, or less than it. On consideration of the question it will be seen that x/ must be greater than 5/, since it is the price of a larger quantity; that is, the fourth term of the proportion is greater than the third. Hence the second term must be greater than the first; therefore 6 lb. must be the second term, and 2 lb. the first.

Second.—Theorem 2 enables us to supply any term of a proportion which may be omitted.

Example.—In the proportion 2 lb : 6 lb. = 5/ : x/, supply the omitted term.

The proportion 2 lb. : 6 lb. = 5/ : x/

may be written $2 : 6 = 5 : x$;

therefore $2 \times x = 6 \times 5$;

therefore $x = \frac{6 \times 5}{2} = 15.$

Hence the omitted term is 15/.

VI. Links.

27. The recommendations under this heading refer more to the spirit than to the substance of the teaching of arithmetic, and definite suggestions must be few. The teacher of arithmetic who knows, at least, algebra and geometry, and is not tied down to the notion that things with different names must be kept severely apart, no matter how close their relation, can always be trusted to teach arithmetic so as to make prominent those general ideas which, though fully developed only in the higher branches of mathematics, are already of fundamental importance in elementary arithmetic.

28. *The Three Fundamental Laws.*—In any attempt to deal with arithmetic as a science, an endeavour should be made to show the pervading importance of the laws of commutation, association, and distribution. With a little skill in the introduction of these notions and a word in season to keep them alive, it is by no means difficult to obtain satisfactory results even with young pupils. The pupils should gain familiarity with these laws as exemplified in addition, subtraction, multiplication, and division. At every stage of an example (say of reducing a particular fractional expression) they should occasionally be required to refer the process to the fundamental law under which it comes.

The development of the science of algebra from these laws has been carried out in the best recent text books, which may be consulted for full details; while the same has been done for arithmetic by Homersham Cox (*Principles of Arithmetic*, Deighton, Bell & Co., Cambridge, 1885).

29. *Generalised Examples.*—It is recommended that examples in arithmetic should be systematically generalised. Thus, following a series of such examples as: If 2 lb. of tea cost 5/ what will 6 lb. cost? there should be given such as: If p lb. of tea cost q/ what will r lb. cost? and then it should be shown how the answer to such a general question contains as particular cases the results of all numerical examples of the same class. Work of this kind ensures the intelligent use of the formulae commonly used in problems of interest and the like.

30. *Mensuration.*—Thoroughly practical examples in mensuration, derived from actual measurement of the school surroundings, with the making of actual plans, can be made a sound basis for geometry, when the pupils come to study it. Whether arithmetical illustrations be part of the teaching of geometry, or geometrical notions be drawn on for examples in arithmetic, is immaterial; but the two subjects should certainly be made to lend mutual aid, as here suggested.

VII. General Hints.

31. Do not hurry from stage to stage in arithmetic at the sacrifice of thoroughness. It is not necessary that a pupil of thirteen should be familiar with stocks and shares.

32. Give many easy examples illustrating principles at the introduction of every new process. This is the best kind of "mental arithmetic."

33. In problems involving a number of multiplications and divisions, insist on having all the steps indicated before any "working out" is attempted. In most cases this leads to great simplification.

34. Encourage the pupil always to make a rough forecast of the result. (To originate this habit it will be found a help to make the pupil write his rough estimate on one side of his slate before proceeding to find the exact result on the other.) The habit of doing this is beneficial in two ways:—

a. It shows the importance of approximate methods generally; and in particular of beginning in multiplication at the left of the multiplier.

b. It will prevent the pupil from resting content with a ridiculous answer.

35. Some method of encouraging speed should be adopted, such as numbering or rearranging the pupils as they finish working. The names of the first half dozen written on the blackboard, even if no additional marks be allowed them, stimulates activity. This plan should, of course, be adopted only at intervals.

36. An attempt should be made to give the pupils some idea of the magnitude of the units of length, weight, &c. Thus, diagrams of the units of the metric system should be placed on the walls of the schoolroom; measurements of the classroom, blackboards, &c., might be made and recorded where they can be readily seen. The weights and dimensions of the slates and the text-books might be ascertained.

37. In practical applications of arithmetic (to problems of money, &c.) some care should be taken that the problems are such as actually do occur. For example, it should be borne in mind that income-tax is never calculated on odd shillings, that the Post Office Savings Bank does not allow interest for a fraction of a month, that banks do not deal in fractions of a penny, and so on.

In teaching "Stocks" the money column of the newspaper should be explained, and be made to furnish problems. It may be mentioned that share lists giving particulars (amount paid up per share, &c.) not contained in the newspapers, can be procured from stock-brokers, or from banks.

For other practical problems geometry and physics should be drawn upon as largely as possible.

38. An operation or a notion should be taught along with its inverse, as, addition with subtraction, multiplication with division, power with root, measure with multiple, &c.

39. As far as exigencies of examinations will allow, postpone formal algebra. The present tendency is to begin algebra too soon by a year or two.

Edinburgh Mathematical Society.

LIST OF MEMBERS.

SIXTH SESSION, 1887–88.

1 Peter Adam, M.A., 81 Claremont Street, Glasgow.

Peter Alexander, M.A., 16 Smith Street, Hillhead, Glasgow.

John Alison, M.A., George Watson's College, Edinburgh (*Hon. Treasurer*).

R. E. Allardice, M.A., F.R.S.E., Assistant to the Professor of Mathematics, University, Edinburgh.

5 P. J. Anderson, M.A., LL.B., Lecturer, Church of Scotland Training College, Aberdeen.

A. H. Anglin, M.A., LL.D., F.R.S.E., M.R.I.A., Professor of Mathematics, Queen's College, Cork.

Paul Aubert, 14 Rue de l'Arsenal, Rochefort-sur-Mer, Charente Inférieure, France.

A. J. G. Barclay, M.A., F.R.S.E., George Watson's College, Edinburgh.

C. W. C. Barlow, M.A., Rochester House, Little Ealing, London, W.

10 Peter Bennett, Glasgow and West of Scotland Technical College, Andersonian Buildings, Glasgow.

James Blyth, M.A., F.R.S.E., Professor of Physics and Mathematics, Andersonian College, Glasgow.

James Bolam, Government Navigation School, Leith.

William Brown, Merchant Taylors' School, Charterhouse, London.

Rev. H. H. Browning, B.D., 128 Byres' Road, Glasgow.

15 James Buchanan, M.A., Peterhouse, Cambridge.

J. R. Burgess, M.A., Mathematical Master, Merchiston Castle, Edinburgh.

John Watt Butters, M.A., 13 M'Dowall Street, Edinburgh.

John Cameron, Mathematical Master, Academy, Montrose.

Peter Campbell, M.A., Musselburgh.

20 Charles Chree, M.A., King's College, Cambridge.

George Chrystal, M.A., LL.D., F.R.S.E., Professor of Mathematics, University, Edinburgh (*Hon. Member*).

John Clark, M.A., Pembroke College, Cambridge.

John Brown Clarke, Heriot's Hospital School, Edinburgh,

Walter Colquhoun, Garnethill School, Glasgow.

25 G. P. Lenox Conyngham, Lieutenant, R.E., India.

James J. Craig, 1 Leven Terrace, Edinburgh.

John Craig, M.A., Heriot's Hospital School, Edinburgh.

Lawrence Crawford, M.A., King's College, Cambridge.

H. G. Creelman, B.A., B.Sc., 7 Parkside Terrace, Edinburgh.

30 J. Crockett, M.A., 10 Gillespie Crescent, Edinburgh.

R. J. Dallas, M.A., King's College, Cambridge.

Hugh Dickie, B.A., Rector, Academy, Kilmarnock.

H. N. Dickson, F.R.S.E., 38 York Place, Edinburgh.

J. D. H. Dickson, M.A., F.R.S.E., Peterhouse, Cambridge.

35 Rev. W. Cooper Dickson, M.A., U.P. Manse, Muckart, by Dollar.

J. Macalister Dodds, M.A., Peterhouse, Cambridge.

Alex. Don, M.A., Waid Academy, Anstruther.

Alex. B. Don, M.A., High School, Dunfermline.

John Don, M.A., Rainings' School, Inverness.

40 John Dougall, M.A., Kippen, Stirling.

Rev. James Edwards, M.A., B.Sc., Professor of Mathematics, Stellenbosch, South Africa.

Arch. C. Elliott, B.Sc., Assistant to the Professor of Engineering, University, Edinburgh.

Jas. Gibson Fairweather, B.Sc., C.E., 8 Findhorn Place, Edinburgh.

R. M. Ferguson, Ph.D., F.R S.E., 8 Queen Street, Edinburgh.

45 Rev. Alex. M. Forbes, M.A, Free Church Manse, Towie, Aberdeenshire.

A. Y. Fraser, M.A., F.R.S.E., Heriot's Hospital School, Edinburgh (*Hon. Secretary*).

John S. Fraser, M.A., Solicitor, 28 High Street, Inverness.

Norman Fraser, M.A., 25 Warrender Park Road, Edinburgh.

E. P. Frederick, M.A., Loretto, Musselburgh.

50 George A. Gibson, M.A., Assistant to the Professor of Mathematics, University, Glasgow (*Vice-President*).

Charles Gordon, F.F.A., F.I.A., Secretary, Scottish Provincial Insurance Company, Union Street, Aberdeen.

R. P. Hardie, 4 Scotland Street, Edinburgh.

William Harvey, B.A., LL.B., 53 Castle Street, Edinburgh.

J. Holliday, 4 Gillespie Crescent, Edinburgh.

55 James Irvine, M.A., B.Sc., 31 Warrender Park Terrace, Edinburgh.

John G. Kerr, M.A., Church of Scotland Training College, Glasgow

Andrew King, M.A., 4 Panmure Place, Edinburgh.

John King, M.A., George Watson's College, Edinburgh.

Cargill C. Knott, D.Sc., F.R.S.E., Professor of Physics, Imperial College, Tokio, Japan.

60 R. Scott Lang, M.A., Professor of Mathematics, University, St Andrews.

A. P. Laurie, B.A., Nairne Lodge, Duddingston.

John Lockie, Consulting Engineer, 2 Custom House Chambers, Leith.

J. Barrie Low, M.A., 17 Elgin Terrace, Dowanhill, Glasgow.

John Mack, M.A., Douglas Cottage, Baillieston, Glasgow.

65 John M'Cowan, M.A., Royal College of Science, Dublin.

Farquhar Macdonald, M.A., High School, Glasgow.

Hector M. Macdonald, M.A., Clare College, Cambridge.

J. C. R. Macdonald, W.S., Dumfries.

W. J. Macdonald, M.A., F.R.S.E., Daniel Stewart's College, The Dean, Edinburgh (*President*).

70 A. Macfarlane, D.Sc., F.R.S.E., University, Austin, Texas, U.S.A.

J. S. Mackay, M.A., LL.D., F.R.S.E., 69 Northumberland Street, Edinburgh.

William Mackay, M.A., Rector, Academy, Rothesay.

J. L. Mackenzie, M.A., Raeden House, Aberdeen.

JOHN MACKENZIE, 3 Valleyfield Street, Edinburgh.

75 P. MACKINLAY, M.A., Rector, Church of Scotland Training College, Edinburgh.

MAGNUS MACLEAN, Natural Philosophy Laboratory, University, Glasgow.

BASIL S. M'LELLAN, M.A., Gordon's College, Aberdeen.

CHARLES M'LEOD, M.A., Assistant to the Professor of Mathematics, University, Aberdeen.

DONALD MACMILLAN, M.A., Clifton Bank School, St Andrews.

80 JAMES MACMILLAN, M.A., 362 St Vincent Street, Glasgow.

J. MACMILLAN, M.A., B.Sc., F.R.S.E., Academy, Perth.

A. J. MACNICOL, M.A., Student, Dunoon.

Rev. J. GORDON MACPHERSON, M.A., Ph.D., F.R.S.E., Ruthven Manse, Meigle, Perthshire.

JOHN M'WILLIAM, M.A., Schoolhouse, Insch.

85 ARTEMAS MARTIN, LL.D., U.S. Coast Survey Office, Washington, D.C., U.S.A.

Rev. CHARLES M. MIDDLETON, M.A., Edinburgh.

W. J. MILLAR, C.E., 261 West George Street, Glasgow.

ANDREW MILLER, M.A., High School, Dundee.

T. HUGH MILLER, M.A., 90 Belmont Gardens, Glasgow.

90 WILLIAM MILNE, M.A., F.R.S.E., High School, Glasgow.

W. T. MITCHELL, M.A., 20 Ampton Street, London, W.C.

ALEXANDER MORGAN, M.A., Church of Scotland Training College, Edinburgh.

J. T. MORRISON, M.A., Heriot-Watt College, Chambers Street, Edinburgh.

ANDREW MUIR, Academy, Glasgow.

95 THOMAS MUIR, M.A., LL.D., F.R.S.E., High School, Glasgow.

R. FRANKLIN MUIRHEAD, M.A., B.Sc., Rochester House, Ealing, London, W.

DAVID MUNN, F.R.S.E., 2 Ramsay Gardens, Edinburgh.

DAVID MURRAY, M.A., B.Sc., High School, Kilmarnock.

CHARLES NIVEN, M.A., D.Sc., F.R.S., Professor of Natural Philosophy, University, Aberdeen (*Hon. Member*).

100 F. GRANT OGILVIE, M.A., B.Sc., F.R.S.E., Principal, Heriot-Watt College, Chambers Street, Edinburgh.

R. T. OMOND, The Observatory, Ben Nevis.

WILLIAM PEDDIE, D.Sc., F.R.S.E., Natural Philosophy Laboratory, University, Edinburgh.

FLORA PHILIP, 8 Dean Terrace, Edinburgh.

ROBERT PHILIP, M.A., Hutcheson's Grammar School, Glasgow.

105 R. H. PINKERTON, Hogganfield, Glasgow.

CHARLES PRENTICE, C.A., 40 Castle Street, Edinburgh.

HARRY RAINY, M.A., 25 George Square, Edinburgh.

WILLIAM RAITT, M.A., B.Sc., 6 Kyle Park, Uddingston.

THOS. T. RANKIN, C.E., Gartsherrie Science Schools, Coatbridge.

110 WILLIAM REID, M.A., High School, Glasgow.

DAVID RENNET, LL.D., 12 Golden Square, Aberdeen.

JAMES RENTON, C.E., 19 Blackwood Crescent, Edinburgh.

WILLIAM RENTON, Ambleside.

D. RINTOUL, M.A., Clifton College, Bristol.

115 Rev. R. S. RITCHIE, Manse of Sorn, Ayrshire.

WILLIAM RITCHIE, George Watson's College, Edinburgh.

ALEX. ROBERTSON, M.A., 30A St Andrew Square, Edinburgh.

ROBT. ROBERTSON, M.A., Ladies' College, Queen St., Edinburgh.

H. C. ROBSON, M.A., 134 Kensington Park Road, London, W.

120 ALEXANDER RUSSELL, M.A., Caius College, Cambridge.

R. F. SCOTT, M.A., St John's College, Cambridge.

WILLIAM SCOTT, M.A., Schoolhouse, New Machar, Aberdeen.

E. J. SMITH, M.A., Royal High School, Edinburgh.

T. B. SPRAGUE, M.A., F.R.S.E., 29 Buckingham Terrace, Edinburgh.

125 G. N. STEWART, M.A., Owen's College, Manchester.

JAMES STRACHAN, M.A., Garnethill School, Glasgow.

P. G. TAIT, M.A., Sec.R.S.E., Professor of Natural Philosophy, University, Edinburgh (*Hon Member*).

JAMES TAYLOR, Academy, Edinburgh.

JAMES TAYLOR, M.A., The Institution, Dollar.

130 GEORGE THOM, M.A., LL.D., The Institution, Dollar.

ANDREW THOMSON, M.A., D.Sc., F.R.S.E., 1 Blackness Crescent, Dundee.

JAMES THOMSON, M.A., The Academy, Ayr.

Sir WILLIAM THOMSON, LL.D., D.C.L., F.R.S.S., L. and E., &c., Professor of Natural Philosophy, University, Glasgow (*Hon. Member*).

RICHARD THORNTON, M.A., Harris Academy, Dundee.

135 DAVID TRAILL, M.A., B.Sc., 14 Livingston Place, Edinburgh.

ROBERT TUCKER, M.A., Hon. Sec. London Mathematical Society, 27 Cantlowes Road, Camden Square, London, N.W.

JOHN E. VERNON, M.A., Arbirlot Free Church Manse, Arbroath.

JAMES WALKER, 3 Warrender Park Terrace, Edinburgh (*last known address*).

SAMUEL WALKER, M.A., B.Sc., Heriot's Hospital School, Edinburgh.

140 JOHN C. WATT, M.A., Jesus College, Cambridge.

JOHN WEIR, M.A., Mathematical Department, University, Glasgow.

WILLIAM WELSH, M.A., Jesus College, Cambridge.

Rev. THOMAS WHITE, B.D., 7 Regent Terrace, Edinburgh.

LAURENCE WIGHT, M.A., Lauder Road, Edinburgh.

145 A. M. WILLIAMS, M.A., 38 Fountainhall Road, Aberdeen.

Rev. JOHN WILSON, M.A., F.R.S.E., 27 Buccleuch Place, Edinburgh.

PHILIP WOOD, M.A., Grammar School, Darlington.

148 WILLIAM WYPER, C.E., 7 Bowmont Gardens, Hillhead, Glasgow.

Proposed at the June Meeting, for Election at the November Meeting, the following :—

D. F. LOWE, M.A., F.R.S.E., Headmaster, Heriot's Hospital School, Edinburgh.

ASUTOSH MUKHOPADHYAY, M.A., F.R.A.S., F.R.S.E., &c., Bhowanipore, Calcutta.

Rev. W. A. REID, M.A., 75 Union Street, Aberdeen.

WILLIAM WILSON, M.A., Heriot's Hospital School, Edinburgh.

Fig 20

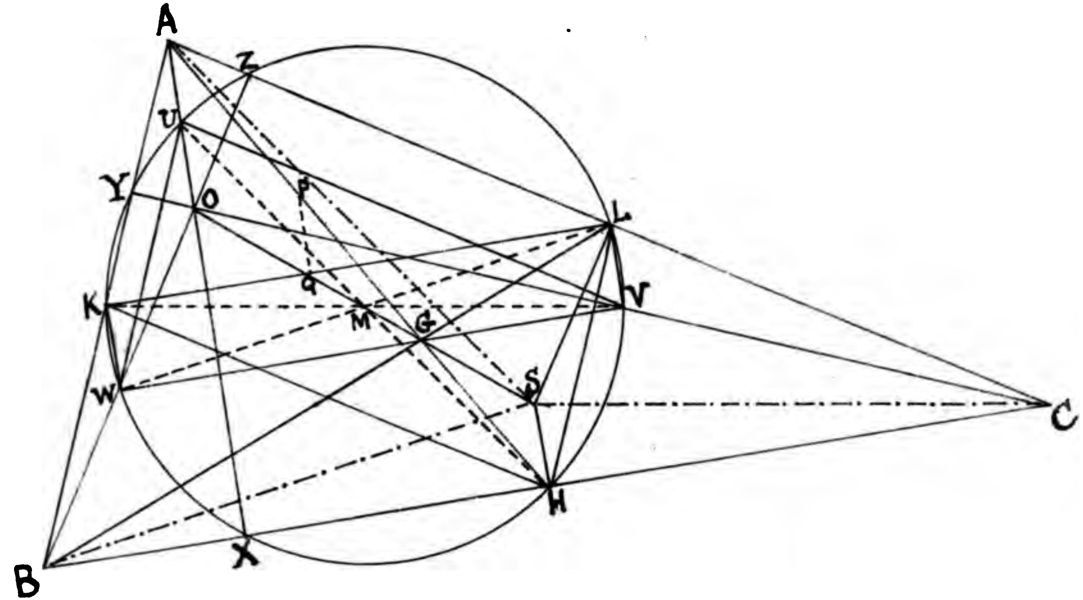


Fig 21

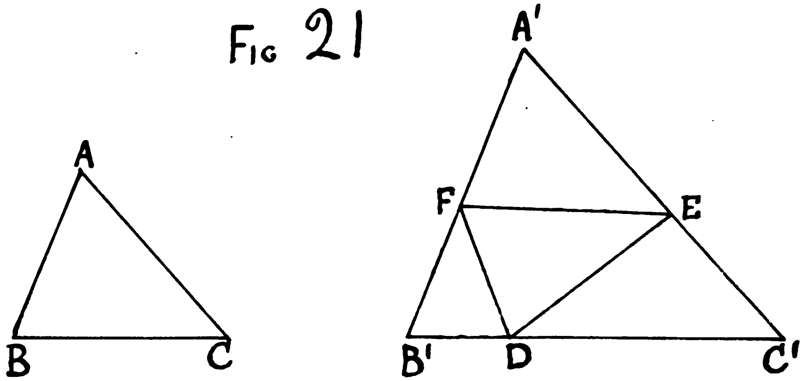


Fig 22

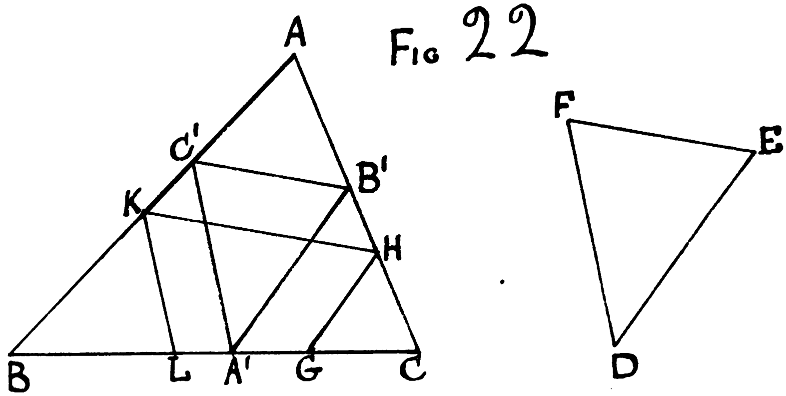


Fig 23

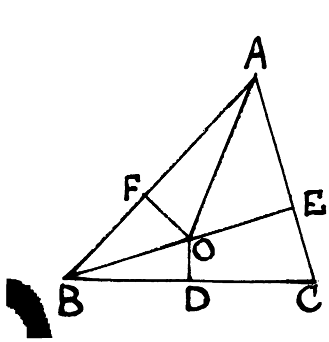


Fig 24

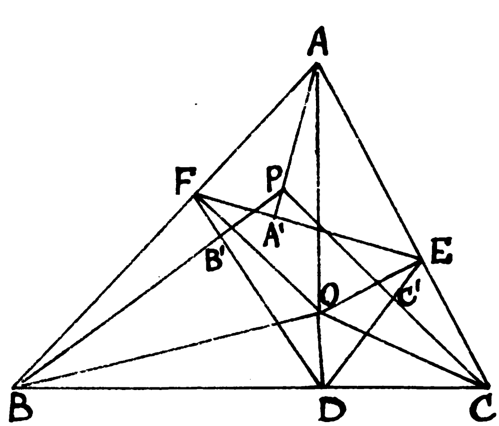


Fig 25

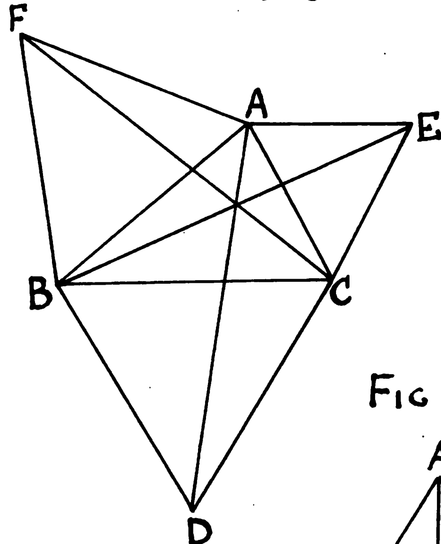


Fig 26

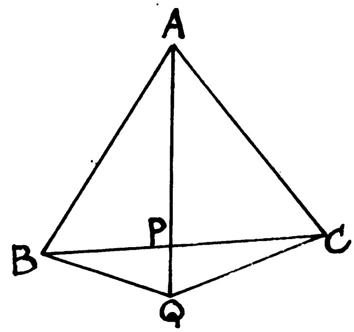


Fig 27

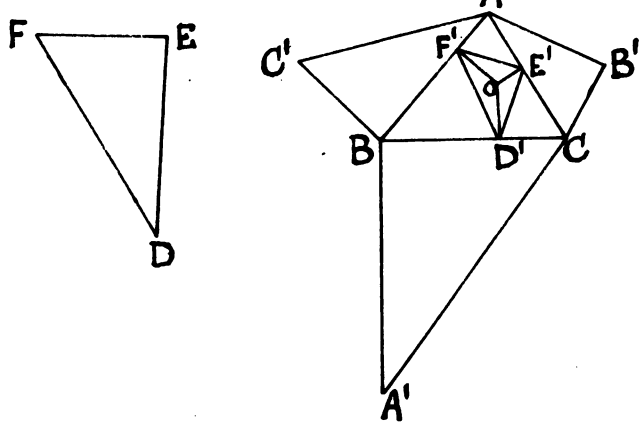


Fig 28

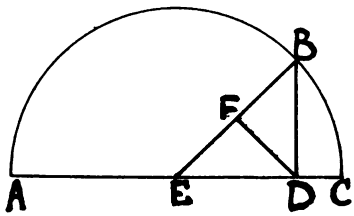


Fig 29

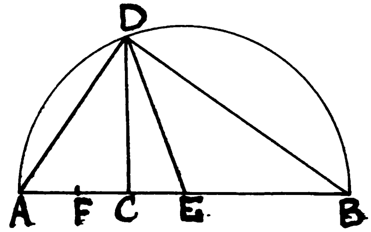


Fig 30

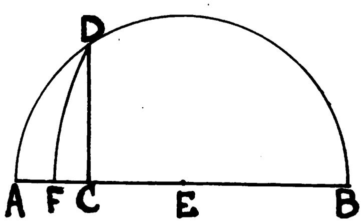


Fig 31

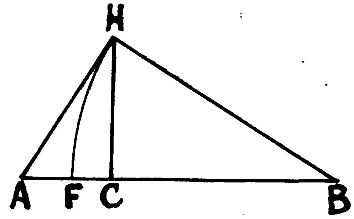


Fig 32

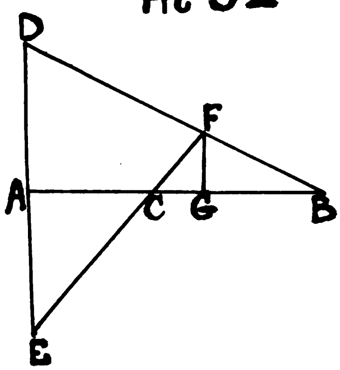


Fig 33

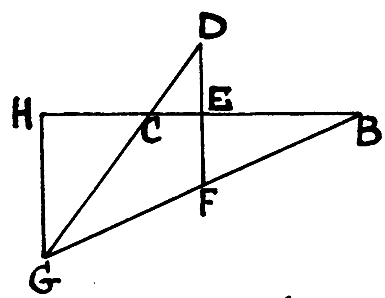


Fig 34

Fig 35

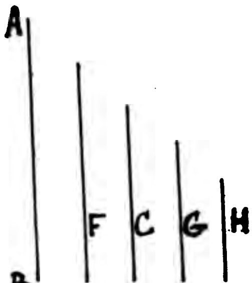


Fig 36

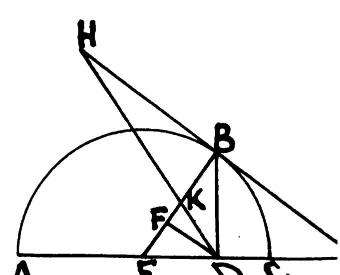


Fig 42

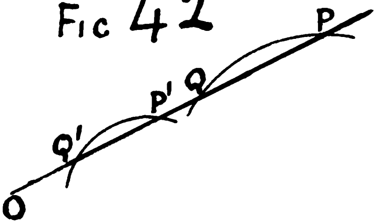


Fig 43

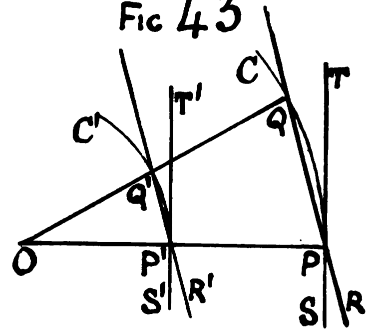


Fig 44

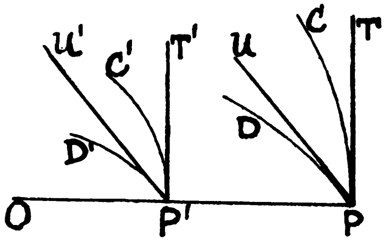


Fig 45

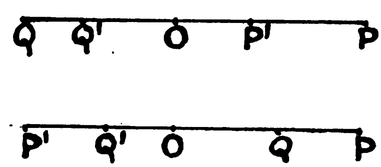


Fig 46

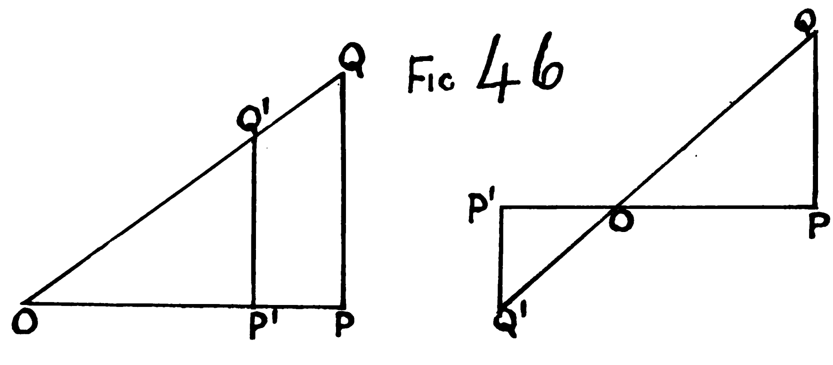


Fig 47

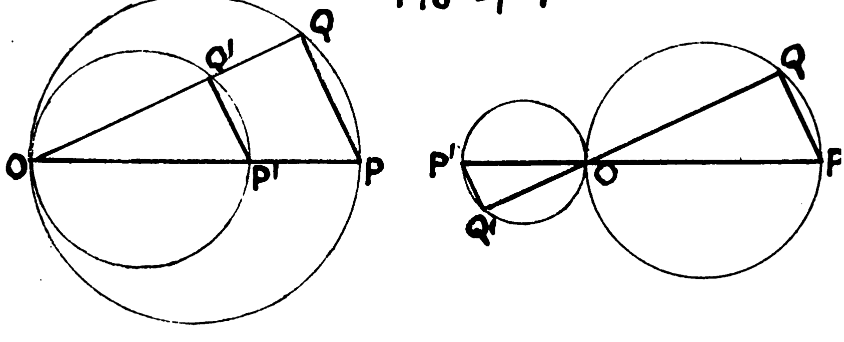


Fig 48

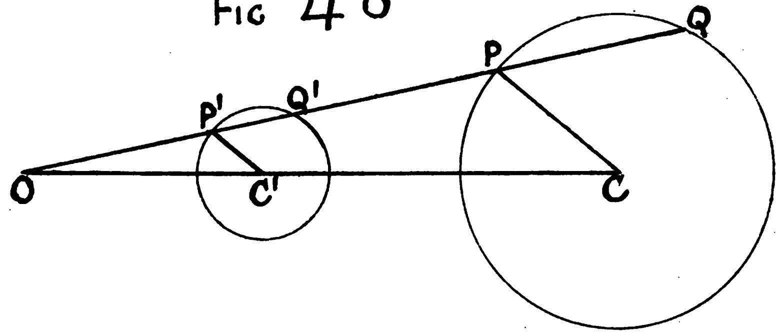


Fig 48

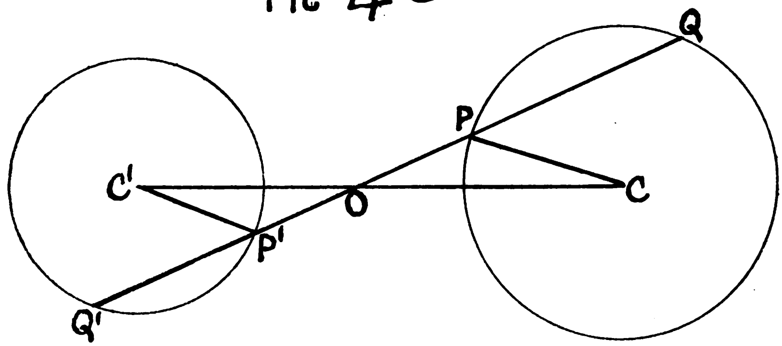


Fig 49

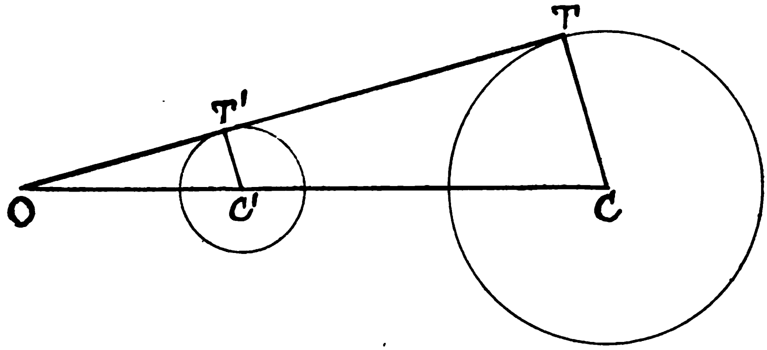


Fig 28

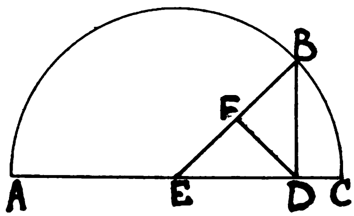


Fig 29

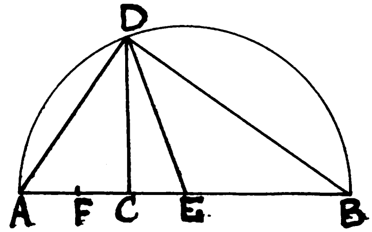


Fig 30

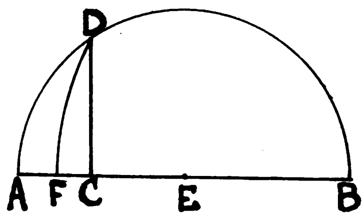


Fig 31

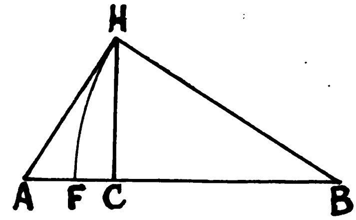


Fig 32

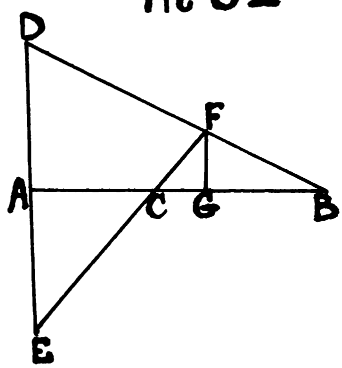


Fig 33

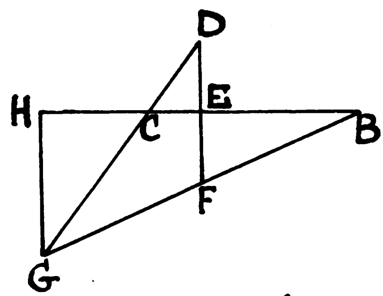


Fig 34

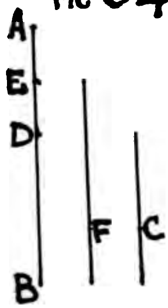


Fig 35

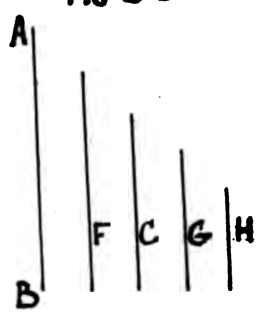


Fig 36

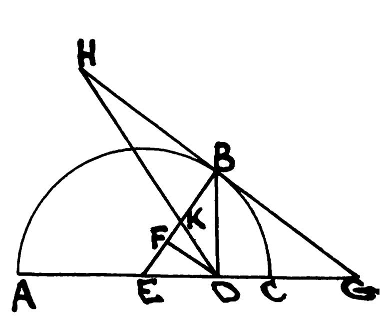


Fig 37

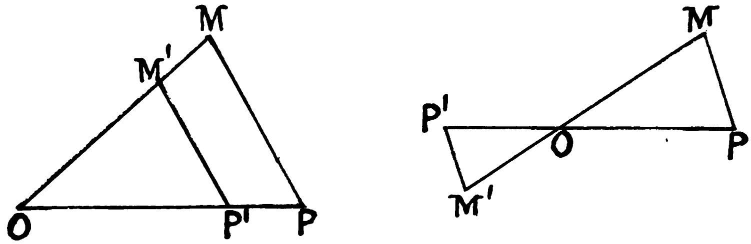


Fig 38

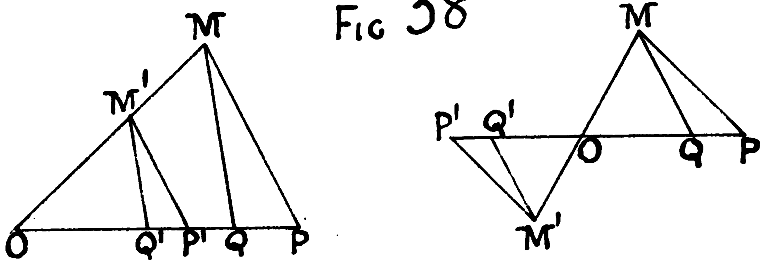


Fig 39

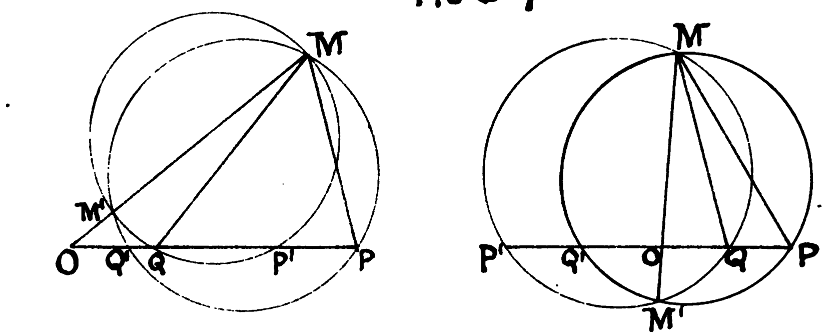


Fig 40

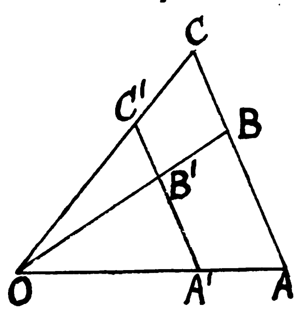


Fig 41

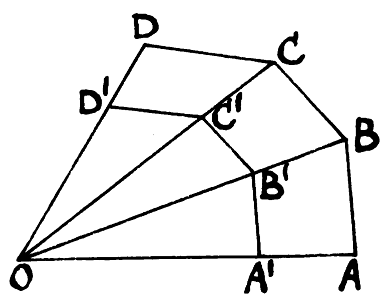


Fig 42

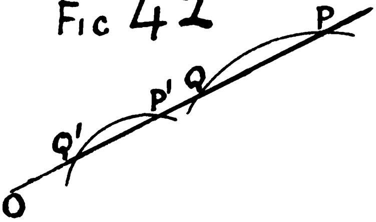


Fig 43

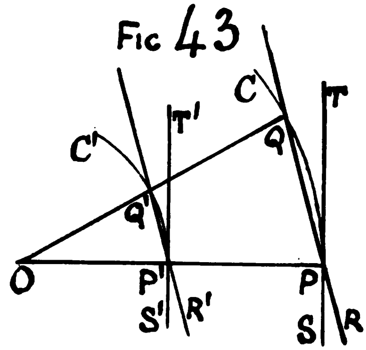


Fig 44

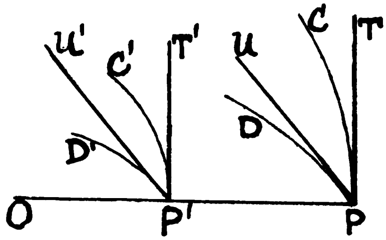


Fig 45

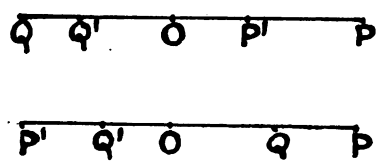


Fig 46

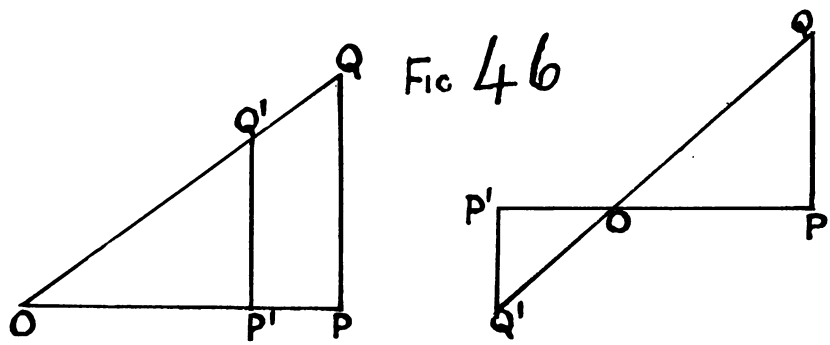


Fig 47

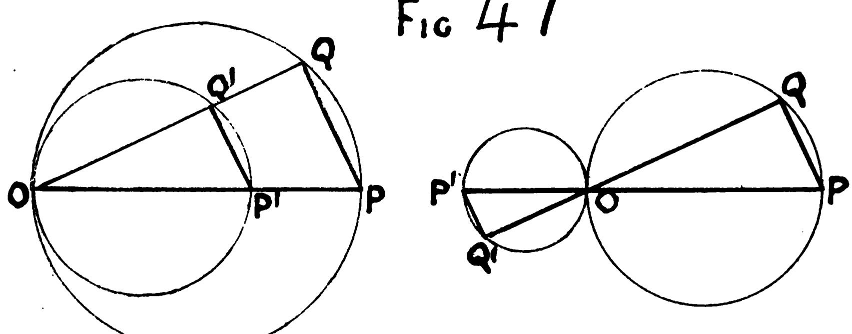


Fig 48

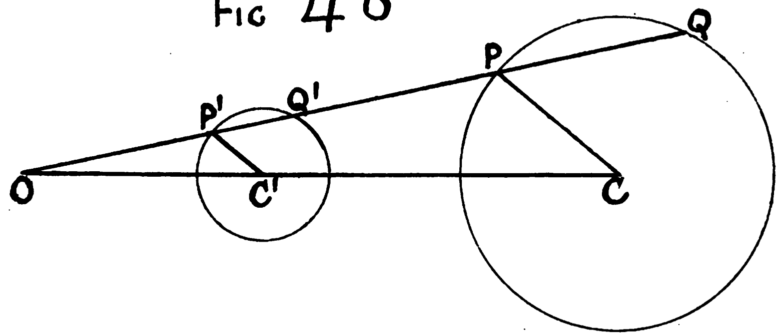


Fig 48

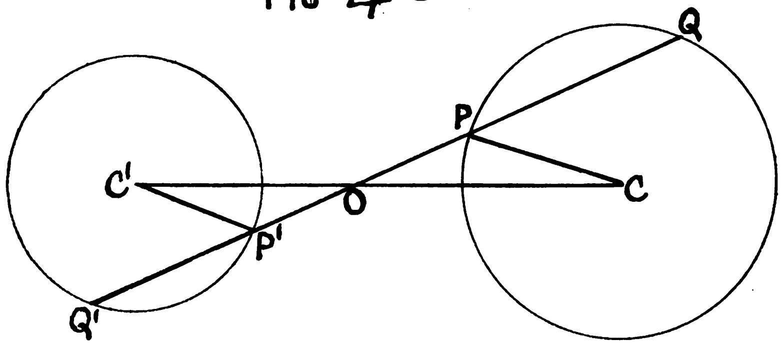


Fig 49